On a Class of Solidarity Values

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Abstract:
We suggest a new one-parameter family of solidarity values for TU-games. The members of this class are distinguished by the type of player whose removal from a game does not affect the remaining players’ payoffs. While the Shapley value and the equal division value are the boundary members of this class, the solidarity value is its center. With exception of the Shapley value, all members of this family are asymptotically equivalent to the equal division value in the sense of Radzik (2013, Math Soc Sci 65, 195-202).
On a class of solidarity values✩

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Abstract

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Keywords: solidarity, null player out, desirability, positivity, asymptotic equivalence

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1. Introduction

A cooperative game with transferable utility (TU-game) consists of a non-empty and finite set of players, \(N\), and a coalition function, \(v : 2^N \rightarrow \mathbb{R}, v(\emptyset) = 0\), which for each coalition \(S \subseteq N\) describes the worth \(v(S)\) that can be generated by its members. Assuming that the grand coalition eventually is formed, the question arises how to distribute the grand coalition’s worth, \(v(N)\). The most prominent single-point solution concept that answers this question probably is the Shapley value (Shapley, 1953).

While almost all modern societies reveal some degree of solidarity, the Shapley value does not allow for solidarity among the players. Unproductive players (null players) are not only assigned zero payoffs, but may even leave the game without affecting the other players’ payoffs. Moreover, a player’s payoff depends only on his own marginal contributions. In contrast, the equal division value distributes the worth generated by the grand coalition equally among the players. This can be interpreted as an extreme kind of solidarity. Clearly,
the payoffs according to the equal division value are almost insensitive to a player’s own marginal contributions.

Several attempts have been made to provide solution concepts that live in between these two extremes. Sprumont (1990) suggests an example of a population monotonic allocation scheme, later on characterized by Nowak and Radzik (1994) as the solidarity value. Nowak and Radzik (1996) study the convex mixtures of the Shapley value and their solidarity value. The class of egalitarian Shapley values, i.e., convex mixtures of the Shapley value and the equal division value is suggested by Joosten (1996). Joosten (1996) and Driessen and Radzik (2002) come up with the discounted Shapley values. The whole class of efficient, linear, and symmetric values is described by Chameni Nembua (2012), for example.

Kamijo and Kongo (2010, 2012) show that the difference between the Shapley value, the equal division value, and the solidarity value that can be pinpointed to just one axiom. The aforementioned values differ in an axiom that specifies the type of player that can be removed from the player set without affecting the remaining players’ payoffs. For the Shapley value, null players can be excluded. Proportional players can be eliminated for the equal division value, where a proportional player is a player whose entrance to a coalition does not change the per capita worth. Closely related is the notion of a quasi-proportional player, which can be removed for the solidarity value.

In this paper, we suggest a family of player types—the $\xi$-players—that contains the null players, the proportional players, and quasi-proportional players. A $\xi$-player is a player whose marginal contribution to a coalition is $\xi$ times the per capita worth of the coalition he enters. The value of $\xi$ may depend on the size of the coalition entered, i.e., $\xi$ actually is sequence of of real numbers. For example, a null player is a $\xi$-player if $\xi$ is constantly zero and a proportional player is a $\xi$-player if $\xi$ is constantly one. The notion of a $\xi$-player gives rise to the corresponding $\xi$-player out axiom. As our first result, we determine those sequences $\xi$ for which there exists a value that satisfies efficiency and the $\xi$-player out axiom. Further, we show that for these sequences there exists a unique value that satisfies efficiency, linearity, symmetry, and the $\xi$-player out axiom.

The family of these values, parametrized by $\xi$, contains values that are not economically sound. In particular, these values may fail desirability (Maschler and Peleg, 1966) or positivity (Kalai and Samet, 1987). Desirability ensures that a player who is more productive than another one does not end up with a lower payoff. Positivity requires that all players obtain non-negative payoffs in monotonic games, i.e., in games, where all marginal contributions are non-negative. Radzik and Driessen (2013) give necessary and sufficient conditions for a general value to satisfy these two properties. As our second result, we employ these conditions to identify the sequences $\xi$ for which the values in our family meet desirability and positivity. The values in this subfamily are called the “generalized solidarity values”. Except the Shapley value and the equal division value, all other egalitarian Shapley values are not members of this subfamily. The solidarity value is the center of the family of generalized solidarity values, which indicates that the name of this class is appropriate.

Radzik (2013) establishes that the solidarity value and the equal division value are asymptotically equivalent in two respects. Let us explain the weaker one. Consider a sequence of games such that the set of players strictly increases, but the worth generated by any
coalition of the these games is bounded. Fix a player in some of these games. Then, the
difference of this player’s payoff for the solidarity value and for the equal division values
converges to zero as the size of the player set goes to infinity. Our third result says that the
generalized solidarity values with exception of the Shapley value, also are asymptotically
equivalent to the equal division value in the sense of Radzik (2013).

The plan of this paper is as follows: In Sect. 2, basic definitions and notation are given. In
Sect. 3, we analyze the player out properties and introduce the generalized solidarity values.
In Sect. 4, we establish the asymptotic equivalence of the generalized solidarity values and
the equal division value. Some remarks conclude the paper. The appendix contains all the
proofs.

2. Basic definitions and notation

Fix a countably infinite set Ω, the universe of players, and let N denote the set of non-
empty and finite subsets of Ω. For R, S, T, N ∈ N, r, s, t, and n denote their cardinalities,
respectively. A (TU)-game is a pair (N,v) consisting of a set of players N ∈ N and a
coalition function v ∈ V(N) := \{ f : 2^N → R | f(∅) = 0 \}, where 2^N denotes the power
set of N. Sometimes, for notational parsimony, we will write v^N instead of v ∈ V(N). Also
for notational convenience, we will write the singleton \{i\} as i.

Subsets of N are called coalitions, and v^N(S) is called the worth of coalition S. For
v,w ∈ V(N), ρ ∈ R, the coalition functions v + w ∈ V(N) and ρ · v ∈ V(N) are given by
(v + w)(S) = v(S) + w(S) and (ρ · v)(S) = ρ · v(S) for all S ⊆ N. For S ⊆ N and
v ∈ V(N), v^|S| ∈ V(S) denotes the restriction of v to 2^S. For T ⊆ N, T ≠ ∅, the game u^T,
u^T_i = 1 if T ⊆ S and u^S_i = 0 otherwise, is called a unanimity game; the game
e^T, e^T_i = 1 if T = S and e^T_i = 0 otherwise, is called a standard game; the game
0^N, 0^N(S) = 0 for all S ⊆ N, is called a null game. Any v ∈ V(N) can be uniquely
represented by unanimity games,

\[ v = \sum_{T ⊆ N : T ≠ ∅} \lambda_T(v) · u^T_i, \quad \lambda_T(v) := \sum_{S ⊆ T : S ≠ ∅}(−1)^{T−S} · v(S). \]  

(1)

Player i ∈ N is called a null player in v ∈ V(N) if v(S ∪ i) = v(S) for all S ⊆ N \ i; players
i,j ∈ N are called symmetric in v ∈ V(N) if v(S ∪ i) = v(S ∪ j) for all S ⊆ N \ {i,j}.

A value on N is an operator ϕ that assigns a payoff vector ϕ(v) ∈ RN to any N ∈ N
and v ∈ V(N). The equal division value is given by

\[ ED_i(v) = \frac{v(N)}{n}, \]  

(2)

for all N ∈ N, v ∈ V(N), and i ∈ N. The Shapley value (Shapley, 1953) is given by

\[ Sh_i(v) = \sum_{S ⊆ N \setminus i} p_{n,s} · (v(S ∪ i) − v(S)) \]  

(3)
for all $N \in \mathcal{N}$, $v \in \mathcal{V}(N)$, and $i \in N$, where

$$p_{n,s} := \frac{1}{n} \cdot \left( \frac{n - 1}{s} \right)^{-1}. \quad (4)$$

The solidarity value (Nowak and Radzik, 1994) is given by

$$\text{So}_i (v) = \sum_{S \subseteq N : i \in S} \frac{p_{n,s-1}}{s} \cdot \sum_{j \in S} (v(S) - v(S \setminus j)) \quad (5)$$

for all $N \in \mathcal{N}$, $v \in \mathcal{V}(N)$, and $i \in N$.

Below, we list the axioms that are referred to later on.

**Efficiency, E.** For all $N \in \mathcal{N}$, $v \in \mathcal{V}(N)$, $\sum_{i \in N} \varphi_i (v) = v(N)$.

**Linearity, L.** For all $N \in \mathcal{N}$, $v, w \in \mathcal{V}(N)$ and $\rho \in \mathbb{R}$, $\varphi(v + w) = \varphi(v) + \varphi(w)$ and $\varphi(\rho \cdot v) = \rho \cdot \varphi(v)$

**Null player, N.** For all $N \in \mathcal{N}$, $v \in \mathcal{V}(N)$ and all $i \in N$ such that $i$ is a null player in $v$, $\varphi_i (v) = 0$.

**Symmetry, S.** For all $N \in \mathcal{N}$, $v \in \mathcal{V}(N)$ and all $i, j \in N$ such that $i$ and $j$ are symmetric in $v$, $\varphi_i (v) = \varphi_j (v)$.

### 3. The family of generalized solidarity values

There is a multitude of values for TU-games on the market. A major purpose of axiomatic characterizations of values is to facilitate the decision which value to apply in a specific situation. Hence, it is of particular interest to pinpoint the difference between values to a single axiom in their characterizations.

Recently, Kamijo and Kongo (2010, 2012) provide characterizations of the Shapley value, the equal division value, and the solidarity value that differ in one axiom only. All characterizations employ the standards axioms efficiency, linearity, and symmetry.\(^1\) The fourth axiom in these characterizations are properties that specify which type of player can leave a game without affecting the remaining players’ payoffs. For the Shapley value, this is the null player.

**Null player out (Derks and Haller, 1999), NPO.** For all $N \in \mathcal{N}$, $v \in \mathcal{V}(N)$, $i \in N$, and $j \in N \setminus i$ such that $i$ is a null player in $v$, $\varphi_j \left( v\mid_{N \setminus i} \right) = \varphi_j (v)$.

For the equal division value and the solidarity value, they introduce the notion of a proportional player and a quasi-proportional player, respectively. Player $i \in N$ is called a proportional player in $v \in \mathcal{V}(N)$ if $v(i) = 0$ and

$$\frac{v(S \cup i)}{s + 1} = \frac{v(S)}{s} \quad \text{for all } S \subseteq N \setminus i, S \neq \emptyset;$$

\(^1\)Actually, they employ the balanced cycle contributions axiom instead of linearity and symmetry. Since the mentioned values satisfy linearity and symmetry, and since by Kamijo and Kongo (2012, Theorem 1), all symmetric and linear values already satisfy balanced cycle contributions, the claim follows.
player \( i \in N \) is called a \textbf{quasi-proportional player} in \( v \in V(N) \) if
\[
\frac{v(S \cup i)}{s+2} = \frac{v(S)}{s+1} \quad \text{for all } S \subseteq N \setminus i.
\]

**Proportional player out, PPO.** For all \( N \in \mathcal{N}, v \in V(N), i \in N, \) and \( j \in N \setminus i \) such that \( i \) is a proportional player in \( v \), \( \varphi_j(v|_{N \setminus i}) = \varphi_j(v) \).

**Quasi-proportional player out, QPO.** For all \( N \in \mathcal{N}, v \in V(N), i \in N, \) and \( j \in N \setminus i \) such that \( i \) is a quasi-proportional player in \( v \), \( \varphi_j(v|_{N \setminus i}) = \varphi_j(v) \).

**Theorem 1 (Kamijo and Kongo, 2012).** (i) The Shapley value is the unique value that satisfies efficiency \((E)\), linearity \((L)\), symmetry \((S)\), and the null player out axiom \((\text{NPO})\).

(ii) The equal division value is the unique value that satisfies efficiency \((E)\), linearity \((L)\), symmetry \((S)\), and the proportional player out axiom \((\text{PPO})\).

(iii) The solidarity value is the unique value that satisfies efficiency \((E)\), linearity \((L)\), symmetry \((S)\), and the quasi-proportional player out axiom \((\text{QPO})\).

Null players, proportional players, and quasi-proportional player are examples of the following class of player types. For any sequence \( \xi:=(\xi_\ell)_{\ell \in \mathbb{N}} \in \mathbb{R}^\mathbb{N} \), a player \( i \in N, N \in \mathcal{N} \) is called a \textbf{\( \xi \)-player} in \( v \in V(N) \), if \( v(i) = 0 \) and
\[
v(S \cup i) - v(S) = \xi_s \cdot \frac{v(S)}{s}
\]
for all \( S \subseteq N \setminus i, S \neq \emptyset \). The sequence \( \xi \) specifies for any coalition size by which share of its per-capita worth a coalition’s worth is changed when a \( \xi \)-player enters a coalition. We employ the notion of a \( \xi \)-player in order to introduce the \( \xi \)-player out axiom below.

**\( \xi \)-player out, \( \xi \)-PO.** For all \( N \in \mathcal{N}, v \in V(N), i \in N, \) and \( j \in N \setminus i \) such that \( i \) is a \( \xi \)-player in \( v \), \( \varphi_j(v|_{N \setminus i}) = \varphi_j(v) \).

Indeed, this class of axioms contains the player out properties above. Let the sequences \( \xi^N, \xi^P, \) and \( \xi^Q \) be given by \( \xi^N_\ell = 0, \xi^P_\ell = 1, \) and \( \xi^Q_\ell = \ell \cdot (\ell + 1)^{-1} \) for all \( \ell \in \mathbb{N} \). One easily checks that a \( \xi^N \)-player is a null player, a \( \xi^P \)-player is a proportional player, and a \( \xi^Q \)-player is a quasi-proportional player, entailing the respective player out properties.

Now, one might be curious which such sequences induce a property that can be satisfied by an efficient value. A sequence \( \xi \in \mathbb{R}^\mathbb{N} \) is called \textbf{admissible} if there exists a value that obeys the \( \xi \)-player out axiom and efficiency. The following theorem identifies the admissible sequences. All proofs are referred to the appendix.

**Theorem 2.** There exists a value that satisfies efficiency \((E)\) and the \( \xi \)-player out property \((\xi \text{-PO})\) if and only if
\[
\xi_1 \in \mathbb{R} \setminus \left\{-\frac{1}{q} \mid q \in \mathbb{N}\right\} \quad \text{and} \quad \xi_\ell = \frac{\ell \cdot \xi_1}{(\ell - 1) \cdot \xi_1 + 1}
\]
for all \( \ell \in \mathbb{N} \).
The theorem implies that the admissible sequences are already determined by their first member. In the following, we restrict attention to admissible sequences. Any admissible sequence $\xi \in \mathbb{R}^N$ is identified by an admissible $\xi := \xi_1 \in \mathbb{R} \setminus \{-\frac{1}{q} \mid q \in \mathbb{N}\}$. From now on, we speak of $\xi$-players and $\xi$-player out axioms ($\xi$-PO). The next theorem shows that any admissible $\xi$ corresponds to a specific value.

**Theorem 3.** For every admissible $\xi$, $\text{So}_\xi$ is the unique value satisfying efficiency (E), linearity (L), symmetry (S), and the $\xi$-player out property ($\xi$-PO). It is given by

$$
\text{So}_\xi (v) = \frac{v(N)}{n} + \sum_{S \subseteq N \setminus i} p_{n,s} \cdot [(1 - \xi_{s+1}) \cdot v(S \cup i) - (1 - \xi_s) \cdot v(S)]
$$

for all $N \in \mathcal{N}$, $v \in \mathcal{V}(N)$, and $i \in N$, where $\xi_\ell = \frac{\ell \xi}{\ell - 1} + 1$ for $\ell \in \mathbb{N}$.\footnote{Note that Theorem 3 provides a separate characterization for every single value $\text{So}_\xi$. In contrast, for example, Weber (1988) suggests characterizations for the whole classes both of semivalues and of probabilistic values.}

Note that Theorem 3 provides a separate characterization for every single value $\text{So}_\xi$. In contrast, for example, Weber (1988) suggests characterizations for the whole classes both of semivalues and of probabilistic values.

An alternative formula for $\text{So}_\xi$ in the spirit of Chameni Nembua (2012) is provided in the following corollary. This formula shows that a player’s payoff depends on both his own marginal contributions and the other players’ marginal contributions. We will use it later in one of the proofs.

**Corollary 4.** For every admissible $\xi$, we have

$$
\text{So}_\xi (v) = \frac{v(i)}{n} + \sum_{S \subseteq N : i \in S, s > 1} p_{n,s-1} \cdot \left(1 - \xi_{s-1}\right) \cdot [v(S) - v(S \setminus i)] \\
+ \xi_{s-1} \cdot \frac{1}{s-1} \sum_{j \in S \setminus i} [v(S) - v(S \setminus j)]
$$

for all $N \in \mathcal{N}$, $v \in \mathcal{V}(N)$, and $i \in N$, where $\xi_\ell = \frac{\ell \xi}{\ell - 1} + 1$ for $\ell \in \mathbb{N}$.

In Theorem 3, linearity and symmetry can be replaced by balanced cycle contributions, which is a weakening of the balanced contributions property\footnote{One easily checks that linearity can be weakened into additivity: For all $N \in \mathcal{N}$, $v, w \in \mathcal{V}(N)$, we have $\varphi(v + w) = \varphi(v) + \varphi(w)$.} due to Myerson (1980), or by differential marginality, which is a differential version of marginality\footnote{Balanced contributions: For all $N \in \mathcal{N}$, $v \in \mathcal{V}(N)$, and $i, j \in N$, $i \neq j$, $\varphi_i(v) - \varphi_i(v|_{N \setminus j}) = \varphi_j(v) - \varphi_j(v|_{N \setminus i})$.} due to Young (1985).
Balanced cycle contributions, BCC (Kamijo and Kongo, 2010). Let \((\mathbb{Z}_n, +)\) denote the cyclic group of order \(n \in \mathbb{N}\). For all \(N \in \mathcal{N}\), \(v \in \mathcal{V}(N)\), and all bijections \(\iota : \mathbb{Z}_{|N|} \rightarrow N\), \(\iota(t) =: t_\iota\), we have

\[
\sum_{t=1}^{n} \left[ \varphi_{t_i}(v) - \varphi_{t_i}(v|_{N \setminus t_{i+1}}) \right] = \sum_{t=1}^{n} \left[ \varphi_{t_j}(v) - \varphi_{t_j}(v|_{N \setminus t_{j-1}}) \right].
\]

Differential marginality, DM (Casajus, 2011). For all \(N \in \mathcal{N}\), \(v, w \in \mathcal{V}(N)\), and \(i, j \in N\) such that

\[
v(S \cup i) - v(S \cup j) = w(S \cup i) - w(S \cup j)
\]

for all \(S \subseteq N \setminus \{i, j\}\), \(\varphi_i(v) - \varphi_j(v) = \varphi_i(w) - \varphi_j(w)\).

Theorem 5. Let \(\xi \in \mathbb{R}^N\) be admissible.

(i) \(\text{So}^\xi\) is the unique value satisfying efficiency (E), the \(\xi\)-player out property (\(\xi\)-PO), and balanced cycle contributions (BCC).

(ii) \(\text{So}^\xi\) is the unique value satisfying efficiency (E), the \(\xi\)-player out property (\(\xi\)-PO), and differential marginality (DM).

The family of values \(\text{So}^\xi\) does not only contain values that express sound solidarity considerations. In particular, these values may not satisfy the following two axioms.

Desirability, D (Maschler and Peleg, 1966). For all \(N \in \mathcal{N}\), \(v \in \mathcal{V}(N)\), and \(i, j \in N\) such that \(v(S \cup i) - v(S) \geq v(S \cup j) - v(S)\) for all \(S \subseteq N \setminus \{i, j\}\), \(\varphi_i(v) \geq \varphi_j(v)\).

Positivity, P (Kalai and Samet, 1987). For all \(N \in \mathcal{N}\), \(v \in \mathcal{V}(N)\), and \(i \in N\) such that \(v\) is monotonic, \(\varphi_i(v) \geq 0\).

Even if players express solidarity among themselves, the payoffs should reflect their individual productivity to some extent. At least, payoff differentials should not be opposite to their productivities. This idea is expressed by the desirability axiom. In monotonic games, no player ever is destructive, i.e., all players always have a non-negative productivity. Hence, even if players show solidarity to less productive ones nobody should end up with a sub-zero payoff. This idea is expressed by the positivity axiom.

The next theorem identifies those values \(\text{So}^\xi\) that meet desirability and positivity. It is an immediate consequence of (7) and Theorem 2 in Radzik and Driessen (2013).

Theorem 6. The value \(\text{So}^\xi\) satisfies desirability (D) and positivity (P) if and only if \(\xi \in [0, 1]\).

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5 Kamijo and Kongo (2010) show that balanced cycle contributions is equivalent to balanced cycle contributions for three player cycles.

6 Note that differential marginality is equivalent to the fairness property due to van den Brink (2001).

7 Isbell (1958) introduced the desirability relation between players in the hypothesis of this axiom for simple games. Desirability is also known as local monotonicity (e.g. Levinsky and Silársky, 2004) or fair treatment (e.g. Radzik and Driessen, 2013).

8 Positivity is also known as monotonicity (e.g. Radzik and Driessen, 2013).
For $\xi \in [0,1]$, we call $\text{So}\xi$ a \textbf{generalized solidarity value}. One easily checks that $\text{So}^0 = \text{Sh}$ and that $\text{So}^1 = \text{ED}$. By Theorem 1 and since an $\frac{1}{2}$-player is a quasi-proportional player, $\text{So}^\frac{1}{2} = \text{So}$.

This already indicates that $\xi$ may determine the degree of solidarity among the players expressed by $\text{So}\xi$, the smallest when $\xi = 0$ and the greatest when $\xi = 1$. For example, by (7), we have

$$\text{So}_i^\xi (e^N_i) = \frac{1 - \xi}{n} (1 - \xi) \cdot \text{Sh}_i (e^N_i)$$

for all $N \in \mathcal{N}$, $|N| > 1$, and $i \in N$. Hence, $\xi \in [0,1]$ can be interpreted as the rate by which player $i$’s Shapley payoff is taxed in $e^N_i$ in order make the other players better off.

A $\xi$-player out axiom postulates that the removal of a $\xi$-player does not affect the other players’ payoffs. Another approach in the literature deals with types of players that receive zero payoffs. In particular, Chameni Nembua (2012) generalizes the null player property by introducing $\alpha$-$A$-null players. For every $(\alpha_s)_{s=2,...,n}$, the corresponding $\alpha$-$A$-null player exhibits zero generalized marginal contributions, i.e., $i$ is an $\alpha$-$A$-null player in $v \in \mathcal{V}(N)$ if $v(i) = 0$ and $A^\alpha_i (v,S) = 0$ for all $S \subseteq N$, $S \ni i$ where

$$A^\alpha_i (v,S) := \alpha_s \cdot [v(S) - v(S \setminus i)] + \frac{1 - \alpha_s}{|S| - 1} \cdot \sum_{j \in S \setminus i} [v(S) - v(S \setminus j)].$$

An $\alpha$-$A$-null player axiom requires an $\alpha$-$A$-null player to obtain zero payoff. Chameni Nembua shows that for a fixed player set $N$, every efficient, linear, and symmetric value is characterized by the aforementioned axioms and a particular $\alpha$-$A$-null player axiom.

This leads to the question, which $\alpha$-$A$-null player axiom characterizes a generalized solidarity value $\text{So}\xi$ for a particular $\xi$. For a given $\xi \in [0,1]$ and a fixed player set $N$, player $i \in N$ is said to be a $\xi$-\textbf{null player} in $v \in \mathcal{V}(N)$ if $v(i) = 0$ and

$$(1 - \xi) \cdot [v(S) - v(S \setminus i)] + \xi \cdot \sum_{j \in S \setminus i} [v(S) - v(S \setminus j)] = 0$$

for all $S \subseteq N \setminus i$. The $\xi$-null player axiom requires a $\xi$-null player to obtain a zero payoff. By Chameni Nembua (2012, Theorem) and Corollary 4, $\text{So}\xi$, $\xi \in [0,1]$ is the unique value on $N$ that satisfies efficiency, linearity, symmetry, and the $\xi$-null player axiom. Clearly, the notions of a $\xi$-player and a $\xi$-null player coincide only for $\xi = 0$. Therefore, the Shapley value is the only efficient, linear, and symmetric value for which the player that receives a zero payoff is the player whose removal leaves the remaining player’s payoffs unaffected, namely, the null player.

4. \textbf{Asymptotic equivalence}

Radzik (2013) shows that the solidarity value is asymptotically equivalent to the equal division value in two regards. In this section, we show that these equivalences also hold for the generalized solidarity values $\text{So}\xi$ for $\xi \in (0,1]$, i.e., whenever a generalized solidarity
value is not the Shapley value. This indicates that the name “generalized solidarity value” for the values $S_{\xi}$ is justified.

Let us start with several definitions introduced in Radzik (2013). In this section, we assume $\mathcal{U} = \mathbb{N}$ and set $N_n := \{1, 2, \ldots, n\}, \mathcal{V}_n = \mathcal{V}(N_n)$, and $\mathcal{V} := \bigcup_{n \in \mathbb{N}} \mathcal{V}_n$. In what follows, all classes of games $\mathcal{V} \subseteq \mathcal{V}$ are such that $\mathcal{V} \cap \mathcal{V}_n \neq \emptyset$ for all $n \in \mathbb{N}$. Further, all values on $N$ considered are linear and efficient. Two values $\varphi$ and $\psi$ are called asymptotically weakly-equivalent in a class of games $\mathcal{V} \subseteq \mathcal{V}$, if for any sequence $(v_n)_{n \in \mathbb{N}}, v_n \in \mathcal{V} \cap \mathcal{V}_n$, we have

$$\lim_{n \to \infty} [\varphi_i (v_n) - \psi_i (v_n)] = 0 \quad \text{for all } i \in \mathcal{U}.$$ 

Two values $\varphi$ and $\psi$ are called asymptotically strongly-equivalent in a class of games $\mathcal{V} \subseteq \mathcal{V}$, if for any sequence $(v_n)_{n \in \mathbb{N}}, v_n \in \mathcal{V} \cap \mathcal{V}_n$ and any sequence $(T_n)_{n \in \mathbb{N}}, T_n \subseteq N_n$, we have

$$\lim_{n \to \infty} \left[ \sum_{i \in T_n} \varphi_i (v_n) - \sum_{i \in T_n} \psi_i (v_n) \right] = 0.$$ 

A class of games $\mathcal{V} \subseteq \mathcal{V}$ is called uniformly bounded, if there is a constant $c \in \mathbb{R}$ such that $|v_n (S)| \leq c$ for all $n \in \mathbb{N}, v_n \in \mathcal{V} \cap \mathcal{V}_n$, and $S \subseteq N_n$. Let $\mathcal{V}_{ug} \subseteq \mathcal{V}$ denote the class of all unanimity games in $\mathcal{V}$.

In particular, Radzik (2013) establishes that (i) the solidarity value and the equal division value are asymptotically weakly-equivalent in any uniformly bounded class $\mathcal{V} \subseteq \mathcal{V}$ and that (ii) the solidarity value and the equal division value are asymptotically strongly-equivalent in the class $\mathcal{V}_{ug} \subseteq \mathcal{V}$ of unanimity games. We extend this result to the generalized solidarity values and partly to the discounted Shapley values (Joosten, 1996; Driessen and Radzik, 2002). The discounted Shapley values, $\text{Shd}^\delta$, $\delta \in [0, 1]$, are given by

$$\text{Shd}^\delta (v) = \sum_{S \subseteq N \setminus i} p_{n,S} \cdot \left[ \delta^{n-s-1} \cdot (v (S \cup i) - \delta \cdot v (S)) \right] \quad (9)$$

for all $N \in \mathcal{N}, i \in N, v \in \mathcal{V} (N)$. Note that $S^1 = \text{Shd}^0 = \text{ED}$ and $S^0 = \text{Shd}^1 = \text{Sh}$.

**Theorem 7.** (i) For $\xi \in (0, 1]$ and $\delta \in [0, 1]$, the generalized solidarity value $S_{\xi}$, the discounted Shapley value $\text{Shd}^\delta$, and the equal division value are pairwise asymptotically weakly-equivalent in any uniformly bounded class $\mathcal{V} \subseteq \mathcal{V}$.

(ii) For $\xi \in (0, 1]$, the generalized solidarity value $S_{\xi}$ and the equal division value are asymptotically strongly-equivalent in the class $\mathcal{V}_{ug} \subseteq \mathcal{V}$ of unanimity games.

(iii) For $\delta \in (0, 1]$, the discounted Shapley value $\text{Shd}^\delta$ and the equal division value are not asymptotically strongly-equivalent in the class $\mathcal{V}_{ug} \subseteq \mathcal{V}$ of unanimity games.

Note that the Shapley value fails both claims. The same holds true for the egalitarian Shapley values $\text{Sh}^\alpha$, $\alpha \in (0, 1]$ introduced by Joosten (1996) given by

$$\text{She}^\alpha (v) = \alpha \cdot \text{Sh} (v) + (1 - \alpha) \cdot \text{ED} (v) \quad (10)$$

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for all $N \in \mathcal{N}$, $v \in \mathcal{V}(N)$. This can be seen as follows:

\[
\lim_{n \to \infty} \text{She}_i^\xi(u_{N_n}^N) \equiv \lim_{n \to \infty} \left( \alpha + \frac{1 - \alpha}{n} \right) = \alpha > 0 \equiv \lim_{n \to \infty} \text{ED}_i(u_{N_n}^N).
\]

While the egalitarian Shapley values fail equivalence even when they are close to the equal division value, the generalized solidarity values exhibits equivalence even when they are close to the Shapley value.

5. Conclusion

In this paper, we generalize two recent contributions to the theory of TU-games. Generalizing the player-out properties of Kamijo and Kongo (2012), we suggest the generalized solidarity values, $\text{So}^\xi$, $\xi \in [0, 1]$, a parametrized class of values that possess a number of desirable properties, in particular, desirability and positivity (Theorems 3 and 6). While the Shapley value ($\text{So}^0$) and the equal division value ($\text{So}^1$) are the boundary values in this class, the solidarity value ($\text{So}^{\frac{1}{2}}$) is exactly at its center. This indicates that the parameter $\xi$ might be interpreted as the degree of solidarity expressed by $\text{So}^\xi$, where a higher $\xi$ stands for more solidarity. In particular, $\xi = 0$ means no solidarity and $\xi = 1$ means comprehensive solidarity. The case that these values indeed are solidarity values is supported by the fact that, for $\xi \neq 0$, $\text{So}^\xi$ exhibits the same asymptotic behavior with respect to equal division value as established by Radzik (2013) for the solidarity value (Theorem 7).

We illustrate both the influence of the parameter $\xi$ and the size of the player set with a numerical example. For $n \in \mathbb{N}$, let $S_n := N_{2n} \setminus N_n$. Figure 1 depicts the payoff $\sum_{i \in S_n} \text{So}_i^\xi(u_{N_n}^{N_{2n}})$
for selected \( n \in \mathbb{N} \). On the one hand, for a given \( n \), the sum of the unproductive players’ payoffs in \( u_{N_n}^{\alpha} \) is positive and strictly increases with the parameter \( \xi \) from 0 (their Shapley payoff) to \( \frac{1}{2} \) (their equal division payoff). This underlines the interpretation of \( \xi \) as a measure of solidarity for \( \text{So}^\xi \). On the other hand, for a given \( \xi \in (0, 1] \), this sum strictly increases in \( n \) and finally approaches \( \frac{1}{2} \), i.e., the equal division payoffs. This reflects our findings on the asymptotic equivalence of the generalized solidarity values for \( \xi \neq 0 \) and the equal division value.

**Appendix A. Proofs**

We prepare the proof of Theorem 2 by a definition and a lemma. For \( N \in \mathcal{N}, T \subseteq N, T \neq \emptyset \), and \( \xi \in \mathbb{R}^N \) define \( u_{\xi,T}^N \) by

\[
u_{\xi,T}^N(S) = \begin{cases} 
\prod_{\ell=1}^{s-1} \left(1 + \frac{\xi_{i}\ell}{\ell}\right), & T \subseteq S, \\
1, & T = S, \\
0, & T \nsubseteq S,
\end{cases} \quad S \subseteq N. \tag{A.1}
\]

**Lemma 8.** The coalition functions defined in (A.1) have the following properties:

(F1) For \( T \subseteq S \subseteq N \), we have \( u_{\xi,T}^N|_S = u_{\xi,T}^S \).

(F2) The players in \( N \setminus T \) are \( \xi \)-players in \( u_{\xi,T}^N \).

(F3) The collection \( \{u_{\xi,T}^N\}_{T \subseteq N, T \neq \emptyset} \) is a basis of \( \mathcal{V}(N) \). For \( v \in \mathcal{V}(N) \), we have

\[
v = \sum_{T \subseteq N : T \neq \emptyset} \lambda_{\xi,T}^N(v) \cdot u_{\xi,T}^N, \tag{A.2}
\]

where

\[
\lambda_{\xi,T}^N(v) = \sum_{S \subseteq T : S \neq \emptyset} (-1)^{t-s} \cdot v(S) \cdot u_{\xi,S}^N(T) \tag{A.3}
\]

for \( T \subseteq N, T \neq \emptyset \).

(F4) For \( T \subseteq S \subseteq N \) and \( v \in \mathcal{V}(N) \), we have \( \lambda_{\xi,T}^N(v) = \lambda_{\xi,T}^S(v|S) \).

(F5) Player \( i \in N \) is a \( \xi \)-player in \( v \in \mathcal{V}(N) \) if and only if \( \lambda_{\xi,T}^N(v) = 0 \) for all \( T \subseteq N \) such that \( i \in T \).
Proof of Lemma 8

(F1) Immediate from (A.1).

(F2) Let \( N \in \mathcal{N}, \ T \subseteq N, \ i \in N \setminus T \) and \( S \subseteq N \setminus i \). By (A.1), \( u^N_{\xi,T}(i) = 0 \). Further, if \( S \neq \emptyset \) and \( T \subseteq S \), then

\[
u^N_{\xi,T}(S) - u^N_{\xi,T}(S) = \prod_{\ell=t}^s \left( 1 + \frac{\xi_{\ell}}{\ell} \right) - \prod_{\ell=t}^{s-1} \left( 1 + \frac{\xi_{\ell}}{\ell} \right) = \frac{\xi_s}{s} \prod_{\ell=t}^{s-1} \left( 1 + \frac{\xi_{\ell}}{\ell} \right) = \frac{\xi_s}{s} \cdot u^N_{\xi,T}(S).
\]

Finally, \( u^N_{\xi,T}(S \cup i) = u^N_{\xi,T}(S) = 0 \) for \( T \not\subseteq S \). That is, \( i \) is a \( \xi \)-player in \( u^N_{\xi,T} \).

(F3) Let \( N \in \mathcal{N} \). Since the dimension of \( \mathcal{V}(N) \) equals the number of members of the set \( B := (u^N_{\xi,T})_{T \subseteq N, T \neq \emptyset} \) it suffices to show that any \( v \in \mathcal{V}(N) \) can be represented as in (A.2) and (A.3). This can be seen as follows. For \( v \in \mathcal{V}(N) \) and \( R \subseteq N \), we have

\[
\left( \sum_{T \subseteq N: T \neq \emptyset} \lambda^N_{\xi,T} (v) \cdot u^N_{\xi,T} \right)(R) \\
= \sum_{T \subseteq N: T \neq \emptyset} \sum_{S \subseteq T: S \neq \emptyset} (-1)^{t-s} \cdot v(S) \cdot u^N_{\xi,S}(T) \cdot u^N_{\xi,T}(R) \\
= \sum_{T \subseteq R: T \neq \emptyset} \sum_{S \subseteq T \subseteq R} (-1)^{t-s} \cdot v(S) \cdot u^N_{\xi,S}(R) \\
= \sum_{S \subseteq R: S \neq \emptyset} \left[ \sum_{t=s}^{r} (-1)^{t-s} \cdot \left( \begin{array}{c} r - s \\ \ell - s \end{array} \right) \right] \cdot v(S) \cdot u^N_{\xi,S}(R) \\
= \sum_{t=0}^{r-s} (-1)^{t} \cdot \left( \begin{array}{c} r - s \\ \ell \end{array} \right) \cdot v(R) \cdot u^N_{\xi,S}(R)
\]

where the last equation drops from (A.1) and \( \sum_{\ell=0}^{r-s} (-1)^{t} \cdot \left( \begin{array}{c} r - s \\ \ell \end{array} \right) = 0 \) for \( r - s \neq 0 \).

(F4) Immediate from F1 and (A.3).

(F5) Sufficiency: Let \( N \in \mathcal{N}, \ v \in \mathcal{V}(N) \), and \( i \in N \) be such that \( (\ast) \lambda^N_{\xi,T}(v) = 0 \) for all
$T \subseteq N$ such that $i \in T$. For $S \subseteq N \setminus i$, we have

$$v(S \cup i) - v(S) \overset{(A.2)}{=} \sum_{T \subseteq N : T \neq \emptyset} \lambda_{\xi, T}^N (v) \cdot [u_{\xi, T}^N (S \cup i) - u_{\xi, T}^N (S)]$$

$$\overset{(A.1), (**)}{=} \sum_{T \subseteq S : T \neq \emptyset} \lambda_{\xi, T}^N (v) \cdot \left[ \prod_{\ell = t}^s \left(1 + \frac{\xi_{\ell}}{s} \right) - \prod_{\ell = t}^{s-1} \left(1 + \frac{\xi_{\ell}}{s} \right) \right]$$

$$= \sum_{T \subseteq S : T \neq \emptyset} \lambda_{\xi, T}^N (v) \cdot \frac{\xi_s}{s} \cdot \prod_{\ell = t}^{s-1} \left(1 + \frac{\xi_{\ell}}{s} \right)$$

$$\overset{(A.1), (A.2)}{=} \xi_s \cdot \frac{v(S)}{s},$$

i.e., $i$ is a $\xi$-player.

Necessity: Let $N \in \mathcal{N}$, $v \in \mathcal{V}(N)$, and $i \in N$ be such that $i$ is a $\xi$-player in $v$. Suppose on the contrary that $\lambda_{\xi, S}^N (v) \neq 0$ for some $S \subseteq N$ such that $i \in S$. (***) Let $S$ be minimal with respect to this property. If $S = i$, then $\lambda_{\xi, S}^N (v) = v(S)$ by (A.3), contradicting $i$ being a $\xi$-player. Let now $|S| > 1$. We have

$$v(S) - v(S \setminus i) \overset{(A.2)}{=} \sum_{T \subseteq N : T \neq \emptyset} \lambda_{\xi, T}^N (v) \cdot [u_{\xi, T}^N (S) - u_{\xi, T}^N (S \setminus i)]$$

$$\overset{(A.1), (A.2)}{=} \lambda_{\xi, S \cup i}^N (v) \cdot u_{\xi, S \cup i}^N (S) + \sum_{T \subseteq S \setminus i : T \neq \emptyset} \lambda_{\xi, T}^N (v) \cdot [u_{\xi, T}^N (S) - u_{\xi, T}^N (S \setminus i)]$$

$$\overset{(A.1), (**)}{=} \lambda_{\xi, S}^N (v) + \xi_{s-1} \cdot \frac{v(S \setminus i)}{s-1},$$

again, contradicting $i$ being a $\xi$-player.

\[ \square \]

Proof of Theorem 2

Let the value $\varphi$ satisfy $\mathbf{E}$ and $\xi$-$\mathbf{PO}$ for $\xi = (\xi_\ell)_{\ell \in \mathbb{N}} \in \mathbb{R}^\mathbb{N}$. Fix $T \in \mathcal{N}$. For $j \in \mathcal{U} \setminus T$, we have

$$\varphi_j \left( u_{\xi, T}^{T \cup j} \right) = u_{\xi, T}^{T \cup j} (T \cup j) - \sum_{i \in T} \varphi_i \left( u_{\xi, T}^{T \cup j} \right)$$

$$\xi_{\mathbf{PO}^{F_1, F_2}} = u_{\xi, T}^{T \cup j} (T \cup j) - \sum_{i \in T} \varphi_i \left( u_{\xi, T}^{T \cup j} \right)$$

$$\mathbf{E} = u_{\xi, T}^{T \cup j} (T \cup j) - 1 \overset{(A.1)}{=} \xi_\ell \frac{t}{t}.$$
Analogously, for \( j, k \in \Omega \setminus T \), one obtains

\[
\phi_j \left( u_{T \cup \{ j, k \}}^{\xi,j} \right) = \left( 1 + \frac{\xi}{t} \right) \cdot \xi^{i+1} \cdot t.
\]

Together with

\[
\phi_j \left( u_{T \cup \{ j, k \}}^{\xi,j} \right) = \xi \cdot \phi_j \left( u_{T \cup \{ j \}}^{\xi,j} \right),
\]

this entails \( \frac{\xi}{t} \neq -1 \) and \( \frac{\xi}{t} = \left( 1 + \frac{\xi}{t} \right)^{-1} \), i.e., \((\xi_i)_{i \in \mathbb{N}}\) is determined by \( \xi_1 \). By induction, one easily shows (6), which proves necessity.

Concerning the sufficiency of the conditions, let \((\xi_i)_{i \in \mathbb{N}}\) satisfy (6). Instead of \( \xi \), we now write \( \xi := \xi_1 \). Note that this implies

\[
u_{\xi,T}^N(S) = 1 + \frac{s - t}{t} \cdot \xi_t, \quad \text{for } T \subseteq S \subseteq N, \ T \neq \emptyset. \quad (A.4)
\]

Define the value \( \phi^\xi \) by

\[
\phi^\xi(v) = \sum_{T \subseteq N \setminus i \in T} \sum_{T \subseteq N \setminus i \in T} \lambda_{\xi,T}^N \left( u_{\xi,T}^N \right) s + \sum_{T \subseteq N \setminus i \in T} \xi_t \cdot \frac{\lambda_{\xi,T}^N (u_{\xi,T}^N)}{s}, \quad (A.5)
\]

for all \( N \in \mathcal{N}, i \in N, \) and \( v \in \mathbb{V}(N) \). Since \( \phi^\xi \) meets L and F3 holds, \( \phi^\xi \) is well-defined and it suffices to show \( E \) for \( u_{\xi,T}^N, \ T \subseteq N, \ T \neq \emptyset \) in order to establish \( E \) in general. We obtain

\[
\sum_{i \in \mathbb{N}} \phi^\xi(u_{\xi,T}^N) = \sum_{i \in \mathbb{N}} \sum_{S \subseteq N \setminus i \in S} \lambda_{\xi,S}^N \left( u_{\xi,S}^N \right) s + \sum_{i \in \mathbb{N}} \sum_{S \subseteq N \setminus i \in S} \xi_t \cdot \frac{\lambda_{\xi,S}^N (u_{\xi,T}^N)}{s}
\]

\[
= \sum_{S \subseteq N \setminus S \neq \emptyset} \lambda_{\xi,S}^N \left( u_{\xi,T}^N \right) s + \sum_{S \subseteq N \setminus S \neq \emptyset} \xi_t \cdot \frac{\lambda_{\xi,S}^N (u_{\xi,S}^N)}{s}
\]

\[
= 1 + (n-t) \cdot \xi_t \cdot \frac{1}{t}, \quad (A.4)
\]

where the third equation employs the fact that \( (u_{\xi,T}^N)_{T \subseteq N, T \neq \emptyset} \) is a basis of \( \mathbb{V}(N) \) and that

\[
(\lambda_{\xi,S}^N \left( u_{\xi,S}^N \right))_{S \subseteq N, S \neq \emptyset}
\]

are the coefficients of \( u_{\xi,T}^N \) with respect to this basis, i.e., \( \lambda_{\xi,T}^N \left( u_{\xi,T}^N \right) = 1 \) and \( \lambda_{\xi,S}^N \left( u_{\xi,S}^N \right) = 0 \) for all \( T, S \subseteq N, T \neq \emptyset, S \neq \emptyset, T \neq S \).

Let \( j \in N \in \mathcal{N} \) be a \( \xi \)-player in \( v \in \mathbb{V}(N) \). For \( i \in N \setminus j \), we have

\[
\phi^\xi_j(v) = \sum_{T \subseteq N \setminus j \setminus i \in T} \lambda_{\xi,T}^N \left( u_{\xi,T}^N \right) s + \sum_{T \subseteq N \setminus \{i,j\} \setminus i \in T} \xi_t \cdot \frac{\lambda_{\xi,T}^N (u_{\xi,T}^N)}{t}, \quad (A.5,F5)
\]

\[
= \sum_{T \subseteq N \setminus j \setminus i \in T} \lambda_{\xi,T}^N \left( u_{\xi,T}^N \right) s + \sum_{T \subseteq N \setminus \{i,j\} \setminus i \in T} \xi_t \cdot \frac{\lambda_{\xi,T}^N (u_{\xi,T}^N)}{t}
\]

\[
= \phi^\xi_j(v|_{N \setminus j}).
\]

Hence, \( \phi^\xi \) meets \( \xi \)-PO. \( \square \)
Proof of Theorem 3

Let $\xi$ be admissible. Existence: By the proof of Theorem 2, $\varphi^\xi$ defined in (A.5) satisfies $E$ and $\xi$-PO. Clearly, $\varphi^\xi$ also satisfies $L$ and $S$.

Uniqueness: Let $\psi$ satisfy $E$, $L$, $S$, and $\xi$-PO. Let $N \in \mathcal{N}$. For $T \subseteq N$, $T \neq \emptyset$, and $i \in T$, we have

$$\psi_i \left( u^N_{\xi,T} \right) \overset{\xi\text{-PO}, \text{F1}}{=} \psi_i \left( u^T_{\xi,T} \right) \overset{(A.1), E, S, \text{1}}{=} \frac{1}{t}.$$  

Now, the payoffs for $j \in N \setminus T$ are given by $E$ and $S$. By $F3$ and $L$, there is at most one such $\psi$.

Correctness of (7): The values $\varphi^\xi$ and $S_0^\xi$ both obey $L$. Fix $N \in \mathcal{N}$. Since $(e^N_R)_{R \subseteq N: R \neq \emptyset}$ is a basis $V(N)$, it suffices to show $\varphi^\xi (e^N_R) = S_0^\xi (e^N_R)$ for all $N \in \mathcal{N}$, $R \subseteq N$, $R \neq \emptyset$. By (A.3) and (A.4), we have

$$\lambda^N_{\xi,T} (e^N_R) = \begin{cases} (-1)^{t-r} \cdot \left( 1 + \frac{t-r}{r} \cdot \xi_r \right), & R \subseteq T, \\ 0, & \text{else.} \end{cases} \quad (A.6)$$

For $i \in R$, one obtains

$$\varphi^\xi_i (e^N_R) \overset{(A.5), (A.6), i \in R}{=} \sum_{R \subseteq T \subseteq N: i \in T} \frac{(-1)^{t-r} \cdot \left( 1 + \frac{t-r}{r} \cdot \xi_r \right)}{t}$$

$$= \sum_{t=r}^n \frac{n-r}{t-r} \cdot \frac{(-1)^{t-r} \cdot \left( 1 + \frac{t-r}{r} \cdot \xi_r \right)}{t}$$

$$= \sum_{t=0}^{n-r} (-1)^t \cdot \frac{n-r}{t} \cdot \frac{1 + \frac{t}{r} \cdot \xi_r}{t + r}$$

$$= (1 - \xi_r) \cdot \sum_{t=0}^{n-r} (-1)^t \cdot \frac{n-r}{t} \cdot \frac{1}{t + r} + \frac{\xi_r}{r} \cdot \sum_{t=0}^{n-r} (-1)^t \cdot \frac{n-r}{t}$$

$$= (1 - \xi_r) \cdot \sum_{t=0}^{n-r} \frac{n-r}{t} \cdot p_{n,r} \overset{(7), i \in R}{=} S_0^\xi (e^N_R). \quad (A.7)$$

The second equality drops from $i \in R$ and the fact that there are $(n-r)$ sets $T$ such that $R \subseteq T \subseteq N$. Now, one obtains the third equation by re-indexing the sum. The fourth equation follows from splitting the sum using the fact that $\frac{1}{t-r} = \frac{1}{t-r} + \frac{1}{t+r}$. The fifth equation drops from (4), $\sum_{t=0}^n (-1)^t \cdot \binom{n}{t} = 0$ for $n \neq 0$, and $\sum_{t=0}^n (-1)^t \cdot \binom{n}{t} \cdot \frac{1}{t+1+m} = \frac{1}{n+m+1} \cdot \binom{n+m}{m}^{-1}$ (for the latter see e.g. Graham et al., 1994, Equation 5.41). Finally,

$$\varphi^\xi_i (e^N_R) \overset{E, S, (A.7), i \in N \setminus R}{=} (1 - \xi_r) \cdot p_{n,r} \overset{(7), i \in N \setminus R}{=} S_0^\xi (e^N_R)$$

for $i \in N \setminus R$. \hfill \square
Proof of Corollary 4

Chameni Nembua (2012) shows that every value \( \varphi \) on \( N \in \mathcal{N} \) satisfying \( \text{E, L, and S}^9 \) is characterized by a sequence \( \alpha = (\alpha_2, \ldots, \alpha_n) \in \mathbb{R}^{n-1} \), such that \( \varphi = \text{CN}^\alpha \), the latter given by

\[
\text{CN}^\alpha (v) = \frac{v(i)}{n} + \sum_{S \subseteq N: i \in S, s > 1} p_{n,s-1} \cdot A^\alpha_i (v, S), \tag{A.8}
\]

for all \( i \in N \) and \( v \in \mathcal{V}(N) \), where

\[
A^\alpha_i (v, S) = \alpha_s \cdot [v(S) - v(S \setminus i)] + \frac{1 - \alpha_s}{s - 1} \cdot \sum_{j \in S \setminus i} [v(S) - v(S \setminus j)].
\]

Casajus and Huettner (2013, Proof of Proposition 1) establish \( \text{CN}^\alpha (e_i^N) = p_{n,t-1} \cdot \alpha_t,1 \) for all \( T \subseteq N, T \neq N \) and \( i \in T \). By (7), we have \( \text{So}^\xi (e_i^N) = p_{n,t-1} \cdot (1 - \xi_t) \) for all \( T \subseteq N, T \neq N \) and \( i \in T \). Further, by Theorem 3, \( \text{So}^\xi \) obeys \( \text{S} \). Hence, \( \text{CN}^\alpha = \text{So}^\xi \) on \( N \) if and only if \( \alpha_t = 1 - \xi_{t-1}, t \in \{2, \ldots, n\} \). \( \square \)

Proof of Theorem 5

(i) By Kamijo and Kongo (2012, Theorem 3), there is at most one value that satisfies \( \text{E, BCC, and } \xi \text{-PO} \) for admissible \( \xi \in \mathbb{R}^N \). Since \( \text{So}^\xi \) is symmetric and linear for admissible \( \xi \in \mathbb{R}^N \), Kamijo and Kongo (2012, Theorem 1) already entail that \( \text{So}^\xi \) satisfies \( \text{BCC} \).

(ii) Let \( \xi \in \mathbb{R} \) be admissible. Existence: By Theorem 3, \( \text{So}^\xi \) meets \( \text{E, L, S, and } \xi \text{-PO} \). In view of van den Brink (2001, Proposition 2.4) and Casajus (2011, Proposition 4), \( \text{L} \) and \( \text{S} \) imply \( \text{DM} \).

Uniqueness: Let the value \( \psi \) obey \( \text{E, DM, and } \xi \text{-PO} \) and let \( N \in \mathcal{N} \). In view of Theorem 3 and Footnote 2, it suffices to show that \( \psi \) satisfies additivity and \( \text{S} \).

Symmetry: Let \( N \in \mathcal{N} \). In \( 0^N \), all players are \( \xi \)-players. By \( \text{E and } \xi \text{-PO} \), we have \((*)\) \( \psi_i (0^N) = 0 \) for all \( i \in N \). Let \( i, j \in N \) by symmetric in \( v \in \mathcal{V}(N) \). This entails

\[
v(S \cup i) - v(S \cup j) = 0 = 0^N (S \cup i) - 0^N (S \cup j)
\]

for all \( S \subseteq N \setminus \{i, j\} \). Hence, we have \( \psi_i (v) - \psi_j (v) \overset{\text{DM}}{=} \psi_i (0^N) - \psi_j (0^N) \overset{(*)}{=} 0 \), i.e., \( \psi \) meets \( \text{S} \).

Additivity: If \( |N| \neq 2 \), one applies Casajus (2011, Proposition 6), which says that \( \text{E, (*)}, \text{and DM} \) imply additivity. Let now \( |N| = 2 \) and \( v_1, v_2 \in \mathcal{V}(N) \). Fix \( k \in N \setminus N \) and let \( v^N \xi k, \ell \in \{1, 2\} \) be given by

\[
v^N \xi k (S) = \begin{cases} v_\ell (S), & S \subseteq N, \\
(1 + \frac{\xi \ell - 1}{s - 1}) \cdot v_\ell (S \setminus k), & k \in S \land |S| > 1,
0, & S \in \{\emptyset, k\}
\end{cases}
\]

\footnote{Instead of symmetry, Chameni Nembua (2012) employs the anonymity axiom: For all \( N \in \mathcal{N}, v \in \mathcal{V}(N), i \in N, \) and all bijections \( \pi : N \rightarrow N, \varphi_{\pi (i)} (N, v \circ \pi^{-1}) = \varphi_i (N, v) \), where \( v \circ \pi^{-1} \in \mathcal{V}(N) \) is given by \( (v \circ \pi^{-1}) (S) = v \left( \pi^{-1} (S) \right) \), \( S \subseteq N \). Yet, Malawski (2008, Theorem 2) shows that efficiency, additivity, and symmetry imply anonymity.}
for all \( S \subseteq N \cup k \). One easily checks that \( k \) is a \( \xi \)-player in \( v_\ell, \ell \in \{1, 2\} \) and in \( v_1 + v_2 \). (**) \( v^{N \cup k}_\ell|_N = v_\ell, \ell \in \{1, 2\} \), and (*** \( (v^{N \cup k}_1 + v^{N \cup k}_2)|_N = v_1 + v_2 \). For \( i \in N \), we have

\[
\psi_i(v_1) + \psi_i(v_2) \overset{(**)}{=} \psi_i(v^{N \cup k}_1) + \psi_i(v^{N \cup k}_2)
\]

\[
\overset{|N \cup k| \neq 2}{=} \psi_i(v^{N \cup k}_1 + v^{N \cup k}_2)
\]

\[
\overset{(**)}{=} \psi_i(v_1 + v_2).
\]

\[\square\]

**Proof of Theorem 7 (i)**

We mimic the proof of Theorem 2 in Radzik (2013). Let \( \mathcal{V} \subseteq \mathcal{V} \) be a uniformly bounded class and let \( c \in \mathbb{R} \) be some bound for \( \mathcal{V} \). For \( \xi = 1 \), nothing is to show. Let now \( \xi \in (0, 1) \). For \( n \in \mathbb{N} \), \( v_n \in \mathcal{V} \cap \mathcal{V}_n \), and \( i \in N_n \), we have

\[
|S_0^\xi_i(v_n)| \leq \xi_n \cdot \frac{|v_n(N_n)|}{n}
\]

\[
+ \sum_{S \subseteq N \setminus i} p_{n,s} \cdot [(1 - \xi_{s+1}) \cdot |v_n(S \cup i)| + (1 - \xi_s) \cdot |v_n(S)|]
\]

\[
\leq \xi_n \cdot \frac{c}{n} + c \cdot \sum_{S \subseteq N \setminus i} p_{n,s} \cdot (1 - \xi_{s+1} + 1 - \xi_s)
\]

\[
= \xi_n \cdot \frac{c}{n} + c \cdot \left(1 - \xi\right) \cdot \sum_{s=0}^{n-1} \left( \frac{1}{s-1 + \frac{1}{\xi}} \right)
\]

\[
= \xi_n \cdot \frac{c}{n} + \frac{2 \cdot c}{n} \cdot \left(1 - \xi\right) \cdot \left(\frac{1}{0 - 1 + \frac{1}{\xi}} + \frac{1}{1 - 1 + \frac{1}{\xi}} + \sum_{s=2}^{n-1} \frac{1}{s - 1 + \frac{1}{\xi}}\right)
\]

\[
\leq \xi_n \cdot \frac{c}{n} + \frac{2 \cdot c}{n} \cdot \left(1 + (1 - \xi)\right) + \frac{2 \cdot c}{n} \cdot \frac{1 - \xi}{\xi} \cdot \sum_{s=1}^{n-1} \frac{1}{s}
\]

where the last but one inequality drops from re-indexing the sum and \( \xi \in (0, 1) \), which implies \( \frac{2 \cdot c}{n} \cdot \frac{1 - \xi}{\xi} > 0 \) and \( \frac{1}{s} > \frac{1}{s + \frac{1}{\xi}} \) for \( s = 1, \ldots, n \).

Since \( \lim_{n \to \infty} \left( \sum_{s=1}^{n} \frac{1}{s} - \ln(n) \right) = \gamma \), where \( \gamma \) is the Euler-Mascheroni constant, and \( \lim_{n \to \infty} \ln(n) = 0 \), we already have \( \lim_{n \to \infty} S_0^\xi_i(v_n) = 0 \). Analogously, one establishes \( \lim_{n \to \infty} \text{Sh}_0^\xi_i(v_n) = 0 \) for \( \delta \in [0, 1) \). In view of the trivial inequality \( \lim \inf_{n \to \infty} |ED_i(v_n)| \leq \lim_{n \to \infty} \frac{\xi}{n} = 0 \), we are done.

\[\square\]
Proof of Theorem 7 (ii)

We will show that, after adopting some necessary changes, the proof of Theorem 4 from Radzik (2013) can be repeated here. Consider sequences \((S_n)_{n \in \mathbb{N}}\), \(\emptyset \neq S_n \subseteq N_n\) and \((T_n)_{n \in \mathbb{N}}\), \(\emptyset \neq T_n \subseteq N_n\). For all \(n, k \in \mathbb{N}\), let \(t_n := |T_n|\), \(s_n := |S_n|\), \(a_n = \frac{t_n}{n}\), and \(b_{n}^{k} = \frac{s_n}{n-k}\).

First, we establish that \(S_{n}^{\xi}\) is asymptotically strongly-equivalent to ED in \(\mathcal{V}_u g\) for \(\xi \in (0, 1]\). Let

\[
A_n := \sum_{j \in T_n} \left[ S_{j}^{\xi} (u_{S_n}^{N_n}) - ED_{j} (u_{S_n}^{N_n}) \right],
\]

\[
B_n := \sum_{j \in N_n \setminus T_n} \left[ S_{j}^{\xi} (u_{S_n}^{N_n}) - ED_{j} (u_{S_n}^{N_n}) \right],
\]

and

\[
g_{n}^{A} := a_n \cdot \left\{ b_{n}^{0} \cdot \left[ 1 + (1 - b_{n}^{0}) + (1 - b_{n}^{2})^2 + \cdots + (1 - b_{n}^{n-s_{n}-1})^{n-s_{n}-1} \right] - 1 \right\}.
\]

What we have to show is \(\lim_{n \to \infty} A_n = 0\). Since the values \(S_{n}^{\xi}\) and ED meet \(E\) and \(S\), \(A_n = B_n = 0\) if \(s_n = n\), and \(A_n + B_n = 0\) in general. Hence, we are allowed to assume that \(s_n < n\) for all \(n \in \mathbb{N}\). Let \(j_n \in N_n \setminus S_n\) for all \(n \in \mathbb{N}\). Moreover, it suffices to show \(\liminf_{n \to \infty} A_n \geq 0\) and \(\liminf_{n \to \infty} B_n \geq 0\).
For $i \in N_n \setminus S_n$, we obtain
\[
S_i \ell (u_{S_n}^N)
\]
\[
\sum_{R \subseteq N : i \notin R, r > 1} p_{n,r-1} \cdot \left( \xi_{r-1} \cdot \frac{1}{r-1} \sum_{j \in R \setminus i} [u_{S_n}^N (R) - u_{S_n}^N (R \setminus j)] \right)
\]
\[
= \sum_{R \subseteq N \setminus i : R \neq \emptyset} p_{n,r} \cdot \left( \xi_r \cdot \frac{1}{r} \sum_{j \in R} [u_{S_n}^N (R \cup i) - u_{S_n}^N ((R \cup i) \setminus j)] \right)
\]
\[
= \sum_{S_n \subseteq R \subseteq N \setminus i : R \neq \emptyset} p_{n,r} \cdot \frac{s_n}{r} \cdot \xi_r
\]
\[
\sum_{r=s_n}^{n-1} \frac{(n-1-s_n)}{n \cdot \binom{n-1}{r}} \cdot \frac{\xi_r \cdot s_n}{r}
\]
\[
= \sum_{r=0}^{n-1-s_n} \frac{(n-1-s_n)}{n \cdot \binom{n-1}{r+1}} \cdot \frac{\xi_{r+s_n} \cdot s_n}{r+s_n}
\]
\[
\sum_{r=0}^{n-1-s_n} \frac{(n-1-s_n)}{n \cdot \binom{n-1}{r+1}} \cdot \frac{(n-1-s_n)!}{r!} \cdot \frac{(r+s_n)!}{(n-1)!} \cdot \frac{1}{(r+s_n-1) \cdot \xi + 1}
\]
\[
= \frac{1}{n} \left[ \frac{1}{(n-2) \cdot \xi + 1} + \frac{n-s_n-1}{n-1} \cdot \frac{1}{(n-3) \cdot \xi + 1} + \frac{(n-s_n-2) \cdot (n-s_n-1)}{(n-2) \cdot (n-1)} \cdot \frac{1}{(n-4) \cdot \xi + 1} + \ldots + \frac{(n-s_n-1)!}{(s_n+1) \ldots (n-1) \cdot \xi + 1} \right],
\]
where the first equation also employs the fact that $i$ is a null player in $u_{S_n}^N$ and the third equation rests on the fact that $u_{S_n}^N (R \cup i) - u_{S_n}^N ((R \cup i) \setminus j) = 0$ whenever $S_n \nsubseteq R$ or $j \notin S_n$. 

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Since $S_n^\xi$ meets $D$, we have

\[
A_n \geq t_n \cdot S_{n_j}^\xi (u_{S_n}^{N_0}) - \frac{t_n}{n} \geq a_n \cdot \left\{ b_0^n \cdot \left[ \frac{n}{(n-2) + \frac{1}{\xi}} \left( 1 + (1 - b_1^n) \cdot \frac{(n-2) + \frac{1}{\xi}}{(n-3) + \frac{1}{\xi}} \right) \right. \\
+ (1 - b_2^n) \cdot (1 - b_3^n) \cdot \frac{(n-2) + \frac{1}{\xi}}{(n-4) + \frac{1}{\xi}} + \ldots \\
+ (1 - b_2^n) \cdot (1 - b_3^n) \ldots \\
\left. \cdot (1 - b_{n-s_n-1}^n) \cdot \frac{(n-2) + \frac{1}{\xi}}{(s_n-1) + \frac{1}{\xi}} \right] - 1 \right\}
\]

\[
\geq a_n \cdot \left\{ b_0^n \cdot \left[ \frac{n}{(n-2) + \frac{1}{\xi}} \left( 1 + (1 - b_1^n) \cdot \frac{(n-2) + \frac{1}{\xi}}{(n-3) + \frac{1}{\xi}} + (1 - b_2^n)^2 \cdot \frac{(n-2) + \frac{1}{\xi}}{(n-4) + \frac{1}{\xi}} \right) \\
+ (1 - b_{n-s_n-1}^n)^{n-s_n-1} \cdot \frac{(n-2) + \frac{1}{\xi}}{(s_n-1) + \frac{1}{\xi}} \right] - 1 \right\}
\]

\[\geq a_n \cdot \left\{ b_0^n \cdot \left[ \frac{n}{(n-2) + \frac{1}{\xi}} \cdot (1 + (1 - b_1^n) \\
+ (1 - b_2^n)^2 + \ldots + (1 - b_{n-s_n-1}^n)^{n-s_n-1} \right] - 1 \right\} \geq g_n^4.
\]

Now, one can exactly repeat the part of the proof of Theorem 4 in Radzik (2013) (starting from formula (32) there) to get $\lim \inf_{n \to \infty} A_n \geq 0$ and $\lim \inf_{n \to \infty} B_n \geq 0$. This completes the proof.

\(\square\)

**Proof of Theorem 7 (iii)**

Let $S_n = N_{2n} \setminus N_n$, $T_n = N_n$, and $i_n \in T_n$. We have

\[
\sum_{i \in T_n} Shd_i^1 (u_{S_n}^{N_{2n}}) = \sum_{i \in T_n} Sh_i (u_{S_n}^{N_{2n}}) = 0 < \frac{1}{2} = \sum_{i \in T_n} ED_i (u_{S_n}^{N_{2n}})
\]

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for all \( n \in \mathbb{N} \). For \( \delta \in (0, 1) \), we obtain

\[
\sum_{i \in T_n} \text{Shd}^\delta_n \left( u_{S_n}^{N_{2n}} \right) = n \cdot \text{Shd}^\delta_n \left( u_{S_n}^{N_{2n}} \right)
\]

\[
\overset{(9)}{=} n \cdot \sum_{R \subseteq N_{2n} \setminus \{i\}} \left[ p_{2n,r} \cdot \delta^{2n-r-1} \cdot (u_{S_n}^{N_{2n}} (R) - \delta \cdot u_{S_n}^{N_{2n}} (R)) \right]
\]

\[
\overset{\text{in} \notin S_n}{=} n \cdot \sum_{S_n \subseteq R \subseteq N_{2n} \setminus \{i\}} \left[ p_{2n,r} \cdot \delta^{2n-r-1} \cdot (1 - \delta) \right]
\]

\[
= n \cdot \sum_{r=n}^{2n-1} \left[ \left( \frac{n-1}{r-n} \right) \cdot p_{2n,r} \cdot \delta^{2n-r-1} \cdot (1 - \delta) \right]
\]

\[
\overset{\text{in} \notin S_n}{=} \frac{1}{2} \sum_{r=n}^{2n-1} \left[ \left( \frac{n-1}{r-n} \right) \cdot \delta^{2n-r-1} \cdot (1 - \delta) \right]
\]

\[
= \frac{1}{2} \left( 1 - \delta + \frac{n-1}{2n-1} \cdot \delta \cdot (1 - \delta) \right) + \frac{1}{2} \sum_{r=0}^{n-3} \left[ \left( \frac{n-1}{r+n} \right) \cdot \delta^{n-r-1} \cdot (1 - \delta) \right]
\]

\[
\overset{\delta \in (0, 1)}{\leq} \frac{1}{2} \left( 1 - \delta + \frac{1}{2} \cdot \delta \cdot (1 - \delta) \right) + \frac{1}{2} \sum_{r=0}^{n-3} \left[ \delta^{n-r-1} \cdot (1 - \delta) \right]
\]

\[
\overset{\delta \in (0, 1)}{=} \frac{1}{2} \left( 1 - \delta + \frac{1}{2} \cdot \delta \cdot (1 - \delta) \right) + \frac{1}{2} \cdot \left( \delta^2 - \delta^n \right)
\]

\[
= \frac{1}{2} \left( 1 - \frac{1}{2} \cdot \delta \cdot (1 - \delta) - \delta^n \right) =: C_n.
\]

Since

\[
\lim_{n \to \infty} C_n = \frac{1}{2} \left( 1 - \frac{1}{2} \cdot \delta \cdot (1 - \delta) \right) < \frac{1}{2} = \lim_{n \to \infty} \sum_{i \in T_n} \text{ED}^\delta_i \left( u_{S_n}^{N_{2n}} \right)
\]

\text{Shd}^\delta \text{ and ED are not asymptotically strongly-equivalent for } \delta \in (0, 1). \]

\section*{References}


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