Abstract:
Within a simple setup, we show that fair redistribution of performance-based allocations implies proportional taxation. Fair redistribution is characterized by three properties: Efficiency, i.e., redistribution has no cost. Anonymity, i.e., rewards after redistribution do not depend on the identity of the members of the society but only on their performance. Weak monotonicity, i.e., whenever both the performance of a certain member of the society as well as the overall performance of the society increase, then this member’s reward after redistribution does not decrease.

This working paper is a corrected version of the HHL Working Paper No. 126.
Fair redistribution of performance-based allocations:
A case for proportional taxation

André Casajus\textsuperscript{a,b,c}

\textsuperscript{a}Chair of Economics and Information Systems, HHL Leipzig Graduate School of Management, Jahnallee 59, 04109 Leipzig, Germany
\textsuperscript{b}LSI Leipziger Spieltheoretisches Institut, Leipzig, Germany
\textsuperscript{c}Institut für Theoretische Volkswirtschaftslehre, Universität Leipzig, Grimmaische Str. 12, 04109 Leipzig, Germany

Abstract

Within a simple setup, we show that fair redistribution of performance-based allocations implies proportional taxation. Fair redistribution is characterized by three properties: Efficiency, i.e., redistribution has no cost. Anonymity, i.e., rewards after redistribution do not depend on the identity of the members of the society but only on their performance. Weak monotonicity, i.e., whenever both the performance of a certain member of the society as well as the overall performance of the society increase, then this member’s reward after redistribution does not decrease.

Keywords: Solidarity, redistribution, proportional taxation, efficiency, anonymity, weak monotonicity

2010 MSC: 91B15  JEL: D63, H20

The moment you abandon the cardinal principle of exacting from all individuals the same proportion of their income or of their property, you are at sea without rudder or compass, and there is no amount of injustice and folly you may not commit.
—McCulloch (1975, p. 174)

1. Introduction

In 1845, McCulloch, author of the most extensive and systematic treatment of public finance in the classical literature, made the above case for proportional taxation. Later on, notable others joined him, for example, Mill (1848), Hayek (1960), Friedman (1962),

\textsuperscript{*}We are grateful to Frank Huettner, Andreas Hoffmann, Enrico Schöbel, and Harald Wiese for insightful comments on this paper.

Email address: mail@casajus.de (André Casajus)
URL: www.casajus.de (André Casajus)

and more recently Hall and Rabushka (1985) and Hall (1996). More formal foundations of proportional taxation have been provided by Moyes and Shorrocks (1998), for example.

In this paper, we make a case for proportional taxation via fair redistribution of performance-based allocations. Particularly, we consider a society in which its members first are rewarded based on their individual contributions to the society’s wealth (individual performance). Modern societies, however, base the allocation of wealth among their members not only on individual performance but also on egalitarian or solidarity principles. This leads to the question of how individual contributions should be redistributed within a society.

We consider three properties of redistribution rules—efficiency, anonymity, and weak monotonicity. For the sake of simplicity, we assume that redistribution has no cost, i.e., redistribution is efficient. Moreover, rewards after redistribution should not depend on the identity of the members of the society but only on their performance, i.e., redistribution is anonymous. Our third property—weak monotonicity—imposes a fairness requirement on redistribution. Whenever both the performance of a certain member of the society as well as the overall performance of the society increase, then this member’s reward after redistribution should not decrease.

For societies comprising more than two members, it turns out that the redistribution rules satisfying these properties are of a particularly simple form. First, the individual performance-based allocations are taxed proportionally at a certain rate. And second, the overall tax revenue is distributed equally within the society.

The next section gives a formal account of this result. Additional technical results can be found in the third section. Some remarks conclude the paper. An appendix contains the proof of our main result.

2. Fair redistribution rules and proportional taxation

In this paper, we consider a particularly simple model of a society. Its members are distinguished only by their individual contributions to the society’s wealth in a certain period of time (for short, performance). In reality, the individual gross income may be viewed as an indicator for these individual contributions. Technically, we consider the $n$-member society $\mathbb{N}_n := \{1, \ldots, n\}$, $n \in \mathbb{N}$, i.e., the society’s members are represented by natural numbers. The individual performances are given by a vector $x \in \mathbb{R}^n$, i.e., we allow for negative performances. This can be justified by considering a period of time in which a member of the society or the society at all does not fare that well.

In real-life societies, members are not just rewarded according to their performances. A considerable amount of the society’s wealth is redistributed via taxation and public spending. In our simple model, this is reflected by redistribution rules. A redistribution rule for an $n$-person society is a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $i \in \mathbb{N}_n$, $f_i(x)$ denotes the reward of member $i$ of the society after redistribution.

---

1 For a general treatment of taxation see Musgrave and Musgrave (1976) or Stiglitz (2000), for example.
In this framework, the desirable properties of redistribution rules advocated in the introduction can be formalized as follows.

**Efficiency, E.** For all \( x \in \mathbb{R}^n \), we have \( \sum_{\ell \in \mathbb{N}_n} f_\ell(x) = \sum_{\ell \in \mathbb{N}_n} x_\ell \), where \( \mathbb{N}_n := \{1, \ldots, n\} \).

The very idea of re-distribution suggests that the sum of individual rewards after redistribution should not be greater than before. In addition, efficiency requires that redistribution comes at no cost. While in real life redistribution is costly, efficiency might be acceptable in our simple and abstract framework.

**Anonymity, An.** For all \( x \in \mathbb{R}^n \), \( i \in \mathbb{N}_n \), and all bijections \( \pi : \mathbb{N}_n \rightarrow \mathbb{N}_n \), we have \( f_{\pi(i)}(\pi x) = f_i(x) \), where \( \pi x \in \mathbb{R}^n \) is given by \( \pi x_{\pi(\ell)} := x_\ell \) for all \( \ell \in \mathbb{N}_n \).

In our setup, the society’s members are fully described by their performance. Therefore, rewards after redistribution should not depend on their identity, i.e., these rewards should not be affected by renaming the members of the society. The mapping \( \pi \) in the above statement of anonymity can be interpreted as a renaming of the society’s members—member \( i \in \mathbb{N}_n \) is assigned the new “name” \( \pi(i) \), which is reflected by setting \( \pi x_{\pi(i)} = x_i \). Anonymity then requires that member \( i \)’s reward after renaming is the same as before, i.e., \( f_{\pi(i)}(\pi x) = f_i(x) \).

Note that anonymity implies that members of the society with the same performance end up with the same reward after redistribution.

**Weak monotonicity, Mo.** For all \( x, y \in \mathbb{R}^n \) and \( i \in \mathbb{N}_n \) such that \( \sum_{\ell \in \mathbb{N}_n} x_\ell \geq \sum_{\ell \in \mathbb{N}_n} y_\ell \) and \( x_i \geq y_i \), we have \( f_i(x) \geq f_i(y) \).

This property relates rewards of a member of the society when individual performances change in a monotonic way. Whenever both the society’s overall performance and the performance of a particular member of the society increase, then this member’s reward after redistribution should not decrease. We feel that a fair redistribution rule should exhibit this property. Increasing overall performance guarantees that no member of the society necessarily has to be rewarded less than before the change. At least, one could argue that a member whose performance does not decrease at the same time actually should not be awarded less.

The following theorem shows that redistribution rules that obey the above three properties entail proportional taxation for societies comprising more than two members, in a sense. Its proof is referred to Appendix A. Note that the one-member case is trivial. We deal with the two-player case in Appendix B.

**Theorem 1.** Let \( n \neq 2 \). A redistribution rule \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) satisfies efficiency (E), anonymity (An), and weak monotonicity (Mo⁻) if and only if there exists some \( \alpha \in [0, 1] \) such that \( f = f^\alpha \), where \( f^\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is given by

\[
f^\alpha_i(x) = \alpha \cdot x_i + (1 - \alpha) \cdot \frac{\sum_{\ell \in \mathbb{N}_n} x_\ell}{n} \quad \text{for all } x \in \mathbb{R}^n \text{ and } i \in \mathbb{N}_n. \tag{1}
\]

The interpretation of formula (1) in terms of proportional taxation transpires by considering the tax rate \( \tau := 1 - \alpha \). This way, member \( i \in \mathbb{N}_n \) keeps the amount of \((1 - \tau) \cdot x_i\)
from his performance, and in addition obtains one nth of the overall tax revenue of \( \tau \cdot \sum_{\ell \in \mathbb{N}_n} x_\ell \). The latter can be interpreted as that all members of the society benefit equally from public spending, which fits our assumption that the society’s members are equal up to performance.

Note that Theorem 1 is silent about the height of the tax rate \( \tau \in [0, 1] \). Fair redistribution only requires proportional taxation at a certain rate. Hence, the height of the tax rate has to be determined by other criteria, for example, optimality considerations. 3

3. Additional technical results

The following lemma shows that anonymity can be relaxed into symmetry in Theorem 1.

**Symmetry, S.** For all \( x \in \mathbb{R}^n \) and \( i, j \in \mathbb{N}_n \) such that \( x_i = x_j \), we have \( f_i (x) = f_j (x) \).

**Lemma 2.** For \( n \neq 2 \), efficiency (E), symmetry (S), and weak monotonicity (Mo) imply anonymity (An).

**Proof.** For \( n = 1 \), nothing is to show. Let now \( n > 2 \) and \( f : \mathbb{R}^n \to \mathbb{R}^n \) exhibit E, S, and Mo. Since any bijection \( \pi : \mathbb{N}_n \to \mathbb{N}_n \) is the finite composition of transpositions, it suffices to show the claim for transpositions. We consider the transposition that interchanges 1 and 2. The proof for the other transpositions runs analogously. Let \( \pi : \mathbb{N}_n \to \mathbb{N}_n \) be such that \( \pi (1) = 2 \), \( \pi (2) = 1 \), and \( \pi (\ell) = \ell \) for all \( \ell \in \mathbb{N}_n \setminus \{1, 2\} \). Let \( y, z \in \mathbb{R}^n \) be given by

\[
y_1 = x_1 \quad \text{and} \quad y_i = \sum_{\ell \in \mathbb{N}_n \setminus \{1\}} \frac{x_\ell}{n-1} \quad \text{for all } i \in \mathbb{N}_n \setminus \{1\} \tag{2}
\]

and

\[
z_2 = x_1 \quad \text{and} \quad y_i = \sum_{\ell \in \mathbb{N}_n \setminus \{1\}} \frac{x_\ell}{n-1} \quad \text{for all } i \in \mathbb{N}_n \setminus \{2\} \tag{3}
\]

We have

\[
f_1 (x) \overset{(2), \text{Mo}}{=} f_1 (y) \tag{4}
\]

\[
f_\pi (x) = f_2 (\pi x) \overset{(3), \text{Mo}}{=} f_2 (z) \tag{5}
\]

\[
f_i (y) \overset{(2), (3), \text{Mo}}{=} f_i (z) \quad \text{for all } i \in \mathbb{N}_n \setminus \{1, 2\} \tag{6}
\]

\[
f_i (y) \overset{(2), \text{S}}{=} f_j (y) \quad \text{for all } i, j \in \mathbb{N}_n \setminus \{1\} \tag{7}
\]

\[
f_i (z) \overset{(3), \text{S}}{=} f_j (z) \quad \text{for all } i, j \in \mathbb{N}_n \setminus \{2\} \tag{8}
\]

Since \( n > 2 \) and by (6), (7), and (8), we have

\[
f_i (y) = f_j (z) \quad \text{for all } i \in \mathbb{N}_n \setminus \{1\} \text{ and } j \in \mathbb{N}_n \setminus \{2\}.
\]

\[\text{For a treatment of optimal taxation see Mirrlees (1971), Atkinson and Stiglitz (1980), and Auerbach (1985), for example.}\]
By (2), (3), and $E$, we obtain $f_{1}(y) = f_{2}(z)$. Moreover, (4) and (5) entail $f_{1}(x) = f_{\pi(1)}(\pi x)$. Analogously, one shows $f_{2}(x) = f_{\pi(2)}(\pi x)$. Finally, $Mo^{-}$ implies $f_{\ell}(x) = f_{\pi(\ell)}(\pi x)$ for all $\ell \in \mathbb{N}_{n} \setminus \{1, 2\}$.  

\begin{remark}
Lemma 2 fails for $n = 2$. Consider the redistribution rule $f^{\triangledown} : \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ given by

$$f_{i}^{\triangledown}(x) = \begin{cases} 
\frac{x_{1} + x_{2}}{2}, & x_{1} \geq x_{2}, \\
\frac{x_{1} - x_{2}}{2}, & x_{1} < x_{2}
\end{cases}$$

for all $x \in \mathbb{R}^{2}$.

It straightforward to show that $f^{\triangledown}$ satisfies $E$, $S$, and $Mo^{-}$ but fails $An$. Note that a third person is needed when Equation (6) is employed in the proof of Lemma 3.

\begin{remark}
The characterizations of the class of redistribution rules $f^{\alpha}$, $\alpha \in [0, 1]$ in Theorem 1 and Corollary 3 are non-redundant for $n > 2$. The redistribution rule $f^{E} : \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ given by $f_{i}^{E}(x) = 0$ for all $x \in \mathbb{R}^{n}$, $i \in \mathbb{N}_{n}$ exhibits all properties but $E$. The redistribution rule $f^{Mo^{-}} : \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ given by $f^{Mo^{-}}(x) = 2 \cdot f^{1}(x) - f^{0}(x)$ for all $x \in \mathbb{R}^{n}$ exhibits all properties but $Mo^{-}$. The redistribution rule $f^{S} : \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ given by $f_{i}(x) = \sum_{\ell \in \mathbb{N}_{n}} x_{\ell}$ and $f_{i}(x) = 0$ for all $x \in \mathbb{R}^{n}$, $i \in \mathbb{N}_{n} \setminus \{1\}$ exhibits all properties but $S$ and $An$.

The class of redistribution rules characterized above contains the egalitarian rule $E := f^{0}$ and performance rule $P := f^{1}$ as polar cases. In fact, $f^{\alpha}$, $\alpha \in [0, 1]$ is the convex mixture the egalitarian rule and the performance rule, $f^{\alpha} = \alpha \cdot P + (1 - \alpha) \cdot E$. In a sense, the same holds true for the monotonicity properties that characterize these rules.

The hypothesis of weak monotonicity consists of two conditions, $\sum_{\ell \in \mathbb{N}_{n}} x_{\ell} \geq \sum_{\ell \in \mathbb{N}_{n}} y_{\ell}$ and $x_{i} \geq y_{i}$. Separating these conditions results in the notions of global monotonicity and strong monotonicity, below. Substituting these properties in Theorem 1 and Corollary 3 for strong monotonicity gives characterizations of the egalitarian rule and of the performance rule, respectively.

Global monotonicity, GMo. For all $x, y \in \mathbb{R}^{n}$ such that $\sum_{\ell \in \mathbb{N}_{n}} x_{\ell} \geq \sum_{\ell \in \mathbb{N}_{n}} y_{\ell}$, we have $f_{i}(x) \geq f_{i}(y)$ for all $i \in \mathbb{N}_{n}$.

Strong monotonicity, Mo. For all $x, y \in \mathbb{R}^{n}$ and $i \in \mathbb{N}_{n}$ such that $x_{i} \geq y_{i}$, we have $f_{i}(x) \geq f_{i}(y)$.

\begin{theorem}
(i) The egalitarian rule is the unique redistribution rule that satisfies efficiency ($E$), symmetry ($S$), and global monotonicity ($GMo$).

(ii) The performance rule is the unique redistribution rule that satisfies efficiency ($E$), symmetry ($S$), and strong monotonicity ($Mo$).
\end{theorem}
Proof. (i) One easily checks that E meets E, S, and GMo. Let now \( f : \mathbb{R}^n \to \mathbb{R}^n \) obey E, S, and GMo. Fix \( x \in \mathbb{R}^n \) and \( i \in \mathbb{N}_n \). Let \( z \in \mathbb{R}^n \) be given by \( z_k = \sum \frac{x_k}{n} \) for all \( k \in \mathbb{N}_n \). By E and S, \( f_i (z) = \sum \frac{z_i}{n} \). Finally, GMo implies \( f_i (x) = f_i (z) \). Hence, \( f = E \).

(ii) One easily checks that P meets E, S, and Mo. Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) obey E, S, and Mo. By E and S, \((*)\) \( f_i (0) = 0 \) for all \( i \in \mathbb{N}_n \), where \( 0 \in \mathbb{R}^n \) is given by \( 0_i := 0 \) for all \( i \in \mathbb{N}_n \). Fix \( x \in \mathbb{R}^n \) and \( i \in \mathbb{N}_n \). Let \( y \in \mathbb{R}^n \) be given by \( y_i = x_i \) and \( y_\ell = 0 \) for all \( \ell \in \mathbb{N}_n \setminus \{i\} \). By \((*)\) and Mo, \( f_i (y) = 0 \) for all \( \ell \in \mathbb{N}_n \setminus \{i\} \). Hence, E entails \( f_i (y) = x_i \). Finally, Mo implies \( f_i (x) = f_i (y) \). Hence, \( f = P \).

4. Concluding remarks

In this paper, we consider the possibly simplest meaningful setup to study redistribution within a society. We show that three intuitive and plausible properties of redistribution rules—efficiency, anonymity, and weak monotonicity—jointly entail proportional taxation of performance-based allocations.

In our simple setup, however, we cannot distinguish between income tax and consumption tax. Hence, Theorem 1 implies proportional overall taxation. In view of the regressive effect usually attributed to consumption tax, our findings do not rule out progressive taxation of income.

Appendix A. Proof of Theorem 1

One easily checks that \( f^\alpha, \alpha \in [0, 1] \) obeys E, An, and Mo\(-\). Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) meet E, An, and Mo\(-\). For \( n = 1 \), the second part of the claim drops from E.

Let now \( n > 2 \). By Mo\(-\), there are mappings \( F_i : \mathbb{R}^2 \to \mathbb{R}, i \in \mathbb{N}_n \) such that

\[
f_i (x) = F_i \left( x_i, \sum_{\ell \in \mathbb{N}_n} x_\ell \right)
\]

for all \( x \in \mathbb{R}^n \) and \( i \in \mathbb{N}_n \). (A.1)

By An, we have \( F_i = F_j =: F \) for all \( i, j \in \mathbb{N}_n \). By E and Mo\(-\), the mapping \( F \) has the following properties.

Efficiency, E\(*\). For all \( a \in \mathbb{R}^n \), we have \( \sum_{\ell \in \mathbb{N}_n} F \left( a_{\ell}, \sum_{\ell \in \mathbb{N}_n} a_{\ell} \right) = \sum_{\ell \in \mathbb{N}_n} a_{\ell} \).

Weak monotonicity, Mo\(*\). For all \( a, a', c, c' \in \mathbb{R} \) such that \( a \geq a' \) and \( c \geq c' \), we have \( F (a, c) \geq F (a', c') \).

For \( c \in \mathbb{R} \), let the mapping \( \Phi_c : \mathbb{R} \to \mathbb{R} \) be given by

\[
\Phi_c (a) := F (a, c) - F (0, c) \quad \text{for all } a \in \mathbb{R}.
\]

For \( a, b, c \in \mathbb{R} \), we have

\[
F (a, c) + F (b, c) + (n - 2) \cdot F \left( \frac{c - a - b}{n - 2}, c \right)
\]

\[
= F (a + b, c) + F (0, c) + (n - 2) \cdot F \left( \frac{c - a - b}{n - 2}, c \right) \quad \text{E\(*\)}.
\]

(A.3)
By (A.2) and (A.3), we obtain
\[ \Phi^c(a) + \Phi^c(b) = \Phi^c(a + b) \quad \text{for all } a, b, c \in \mathbb{R}. \]  
(A.4)

This already entails
\[ \Phi^c(\rho \cdot a) = \rho \cdot \Phi^c(a) \quad \text{for all } a, c \in \mathbb{R} \text{ and } \rho \in \mathbb{Q}. \]
(A.5)

By Mo\(^*\), \( \Phi^c \) is monotonic, i.e., \( \Phi^c(a) \geq \Phi^c(b) \) for all \( a, b \in \mathbb{R} \) such that \( a \geq b \). Since \( \mathbb{Q} \) is a dense subset of \( \mathbb{R} \), (A.5) entails
\[ \Phi^c(\rho \cdot a) = \rho \cdot \Phi^c(a) \quad \text{for all } a, c, \rho \in \mathbb{R}. \]  
(A.6)

For all \( c \in \mathbb{R} \), set
\[ \alpha_c := F(1, c) - F(0, c). \]  
(A.7)

By (A.2), (A.6), and (A.7), we have
\[ F(a, c) = \alpha_c \cdot a + F(0, c) \quad \text{for all } a, c \in \mathbb{R}. \]  
(A.8)

Moreover, we obtain
\[ \frac{c}{n} \in \mathbb{E}^* \quad \text{(A.8)} \]
\[ = F\left( \frac{c}{n}, c \right) = \alpha_c \cdot \frac{c}{n} + F(0, c), \]

i.e., \( F(0, c) = (1 - \alpha_c) \cdot \frac{c}{n} \) for all \( c \in \mathbb{R} \) and therefore
\[ F(a, c) = \alpha_c \cdot a + (1 - \alpha_c) \cdot \frac{c}{n} \quad \text{for all } a, c \in \mathbb{R}. \]  
(A.9)

Now, we show that \( \alpha_c \) does not depend on \( c \). Let \( c, c' \in \mathbb{R}, \ c > c' \). Suppose \( \alpha_c \neq \alpha_{c'} \). By (A.9), we have
\[ F(a, c) - F(a, c') = (\alpha_c - \alpha_{c'}) \cdot a + (1 - \alpha_c) \cdot \frac{c}{n} - (1 - \alpha_{c'}) \cdot \frac{c'}{n} \quad \text{for all } a \in \mathbb{R}, \]
i.e., one can find some \( a^* \in \mathbb{R} \) such that \( F(a^*, c') > F(a^*, c) \), contradicting Mo\(^*\). Thus, \( \alpha_c = \alpha_{c'} =: \alpha \) for all \( c, c' \in \mathbb{R} \).

By (A.7) and Mo\(^*\), we have \( \alpha \geq 0 \). Moreover, we have
\[ 0 = F(0, 0) \leq F(0, 1) = \frac{1 - \alpha}{n}, \]
i.e., \( 1 \geq \alpha \). Finally, by (A.1), (A.9), (1), and our findings on \( \alpha \), we have \( f = f^\alpha \) for some \( \alpha \in [0, 1] \). \( \square \)
Appendix B. Societies with two members

Theorem 1 fails for \( n = 2 \). To see this, consider the redistribution rule \( f^\triangledown \) for \( \mathbb{N}_2 \) given by

\[
(f_1^\triangledown (x), f_2^\triangledown (x)) = \begin{cases} 
(x_1, x_1), & \text{(i) } x_1 \geq 0 \land x_2 \geq 0, \\
(0, x_1 + x_2), & \text{(iia) } x_1 < 0 \land x_2 > 0 \land x_1 + x_2 \geq 0, \\
(x_1 + x_2, 0), & \text{(iib) } x_1 < 0 \land x_2 > 0 \land x_1 + x_2 < 0, \\
(x_1, x_2), & \text{(iii) } x_1 \leq 0 \land x_2 \leq 0, \\
(0, x_1 + x_2), & \text{(iva) } x_1 > 0 \land x_2 < 0 \land 0 \geq x_2 + x_2, \\
(x_1 + x_2, 0), & \text{(ivb) } x_1 > 0 \land x_2 < 0 \land x_1 + x_2 > 0
\end{cases}
\]  

(B.1)

for all \( x \in \mathbb{R}_+^2 \). It is straightforward to show that \( f^\triangledown \) meets \( \mathbf{E} \) and \( \mathbf{An} \). Remains to show that \( f^\triangledown \) also satisfies \( \mathbf{Mo}^- \). Let (*) \( x, y \in \mathbb{R}^2 \) and \( i \in N \) be as in the hypothesis of \( \mathbf{Mo}^- \). By \( \mathbf{An} \), we are allowed to assume \( i = 1 \) without loss of generality. Moreover, the cases below are exhaustive and consistent given (*).

1. As long as \( x \) and \( y \) stay within the same subcase of (B.1), the implication of \( \mathbf{Mo}^- \) holds true.
2. If \( y \) is as in (iia), (iib), or (iii) and \( x \) as in (i), (iva), or (ivb), then \( f_1^\triangledown (y) \leq 0 \leq f_1^\triangledown (x) \).
3. If \( y \) as in (iib) and \( x \) as in (iia), then \( f_1^\triangledown (y) = y_1 + y_2 < 0 = f_1^\triangledown (x) \).
4. If \( y \) as in (iib) and \( x \) as in (iii), then \( f_1^\triangledown (y) = y_1 + y_2 < y_1 \leq x_1 = f_1^\triangledown (x) \).
5. If \( y \) as in (iii) and \( x \) as in (iib), then \( f_1^\triangledown (y) = y_1 \leq x_1 \leq x_1 + x_2 = f_1^\triangledown (x) \).
6. If \( y \) as in (iii) and \( x \) as in (iia), then \( f_1^\triangledown (y) = y_1 \leq 0 = f_1^\triangledown (x) \).
7. If \( y \) as in (iva) and \( x \) as in (ivb), then \( f_1^\triangledown (y) = 0 \leq x_1 + x_2 = f_1^\triangledown (x) \).
8. If \( y \) as in (iva) and \( x \) as in (i), then \( f_1^\triangledown (y) = 0 \leq x_1 = f_1^\triangledown (x) \).
9. If \( y \) as in (ivb) and \( x \) as in (i), then \( f_1^\triangledown (y) = y_1 + y_2 \leq y_1 \leq x_1 = f_1^\triangledown (x) \).
10. If \( y \) as in (i) and \( x \) as in (ivb), then \( f_1^\triangledown (y) = y_1 \leq y_1 + y_2 \leq x_1 + x_2 = f_1^\triangledown (x) \).

Theorem 1 holds for \( n = 2 \), if one adds the following property of distribution rules.

\textbf{Linearity, L.} For all \( x, y \in \mathbb{R}^n \) and \( \rho \in \mathbb{R} \), we have \( f(x+y) = f(x) + f(y) \) and \( f(\rho \cdot x) = \rho \cdot f(x) \).

\textbf{Proposition 7.} A redistribution rule \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) satisfies efficiency (\( \mathbf{E} \)), linearity (\( \mathbf{L} \)), anonymity (\( \mathbf{An} \)), and weak monotonicity (\( \mathbf{Mo}^- \)) if and only if there exists some \( \alpha \in [0, 1] \) such that \( f = f^\alpha \).

\textbf{Proof.} One easily checks that \( f^\alpha, \alpha \in [0, 1] \) obeys \( \mathbf{E}, \mathbf{L}, \mathbf{An}, \) and \( \mathbf{Mo}^- \). Let the redistribution rule \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) satisfy \( \mathbf{E}, \mathbf{L}, \mathbf{An}, \) and \( \mathbf{Mo}^- \). By \( \mathbf{An} \), we have

\[
f_1(0, 1) = f_2(1, 0)
\]

(B.2)

Set

\[
\alpha := f_1(1, 0) - f_2(1, 0) \overset{(B.2)}{=} f_1(1, 0) - f_1(0, 1)
\]

(B.3)
By \( \textbf{Mo}^- \), we have \( f_1 (1, 0) \geq f_1 (0, 1) \), i.e., \( \alpha \geq 0 \). By \( \textbf{E} \) and \( \textbf{An} \), one obtains \( f_2 (0, 0) = 0 \). Hence, \( \textbf{Mo}^- \) entails \( f_2 (1, 0) \geq 0 \). By (B.3) and \( \textbf{E} \), we therefore have \( \alpha = 1 - 2 \cdot f_2 (1, 0) \leq 1 \). All in all, \( \alpha \in [0, 1] \).

By \( \textbf{E} \), we have
\[
f_1 (1, 0) + f_2 (1, 0) = 1. \tag{B.4}
\]
Solving (B.2), (B.3), and (B.4) for \( f_1 (0, 1) \) gives
\[
f_1 (0, 1) = \frac{1 - \alpha}{2}. \tag{B.5}
\]
Finally, we obtain
\[
f_1 (x) \stackrel{!}{=} x_1 \cdot f_1 (1, 0) + x_2 \cdot f_1 (0, 1) \stackrel{(B.3),(B.5)}{=} \alpha \cdot x_1 + (1 - \alpha) \cdot \frac{x_1 + x_2}{2} \stackrel{(1)}{=} f_1^\alpha (x)
\]
for all \( x \in \mathbb{R}^2 \). Since both \( f \) and \( f^\alpha \) meet \( \textbf{E} \), we also have \( f_2 (x) = f_2^\alpha (x) \) for all \( x \in \mathbb{R}^2 \), i.e., \( f = f^\alpha \).

\[
\square
\]

References


