Monotonic Redistribution of Non-Negative Allocations

Another Case for Proportional Taxation

André Casajus

PD Dr. André Casajus is a research associate at the Chair of Economics and Information Systems at HHL Leipzig Graduate School of Management, Germany. Email: andre.casajus@hhl.de

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We reconsider Casajus’ (2014, Theoretical Economics, forthcoming) characterization of proportional taxation by three properties of redistribution: efficiency, symmetry, and monotonicity. When restricted to non-negative allocations, these properties only imply proportional taxation in a weaker sense - the tax rate may vary with overall performance but not in an arbitrary fashion. On the restricted domain, proportional taxation is characterized by the afore-mentioned properties together with positive homogeneity, i.e., upscaling performances implies upscaled rewards after redistribution.
Monotonic redistribution of non-negative allocations: another case for proportional taxation✩

André Casajus

*Economics and Information Systems, HHL Leipzig Graduate School of Management
Jahnallee 59, 04109 Leipzig, Germany

Abstract
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1. Introduction

The above quotation from the King James Bible provides an early example of the idea of proportional taxation. In modern times, proportional taxation has been advocated by notable people like McCulloch (1845), Mill (1848), Hayek (1960), and Friedman (1962), or later by Hall and Rabushka (1985) and Hall (1996). Only recently, Casajus (2014) makes an axiomatic case for proportional taxation1 via monotonic redistribution of performance-based allocations.2

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Email address: mail@casajus.de (André Casajus)
URL: www.casajus.de (André Casajus)

1Fleubaey and Maniquet (2011, Chapters 10 and 11), for example, provide a survey of axiomatic foundations of taxation.


In particular, Casajus considers redistribution in a very simple model of a society where its members are characterized by their individual contributions to the society’s wealth. A redistribution rule assigns to any list of individual performance-based allocations (for short, performances) a list of allocations after redistribution. Cum grano salis\textsuperscript{3}, three properties of redistribution rules—efficiency, symmetry, and monotonicity\textsuperscript{4}—entail proportional taxation in the following sense. First, individual performances are taxed proportionally at a certain rate. And second, overall tax revenue is distributed equally within the society.

A peculiarity of this model is that individual performances may be negative. A fortiori, Casajus’ proof critically makes use of negative individual performances. This can be justified when one considers redistribution within a certain period of time where individuals may incur losses. On the one hand, it is possible to interpret the taxation of losses. In many tax systems, losses in one tax period may reduce the tax burden in future periods. Taxation of losses then means that these reductions are accounted in the period where the losses accrued. On the other hand, one would like to have a justification for proportional taxation of non-negative performances in the same vein as for the full domain.

Bad tidings is that the restriction of Casajus (2014, Theorem 1) to non-negative performances does not hold true. However, things aren’t too bad. Within the restricted domain, the three properties still imply a weaker version of proportional taxation (Theorem 2). For any fixed overall performance, redistribution takes place by proportional taxation. Yet, the tax rate may vary with overall performance but not in an arbitrary fashion. In particular, overall tax revenue must not decrease when overall performance increases, i.e., tax rate is not allowed to drop too fast. Moreover, overall tax revenue on a fixed overall performance cannot increase more than per-capita tax revenue with increasing overall performance, i.e., tax rate cannot rise too fast. While the former requirement “protects” members of the society with a zero performance, the latter one “protects” a member who is the sole performer in the society. Technically, these conditions can be expressed by a lower and upper bound on the elasticity of the tax rate function that essentially has to be absolutely continuous (Theorem 4).

Our counterexample on the restricted domain fails homogeneity for positive scalars, i.e., scaled up performances may be taxed at a different rate. It turns out that adding positive homogeneity to the list of properties yields a characterization of proportional taxation (Theorem 5).

The next section gives a formal account and discussion of these results. Some remarks conclude the paper. Two appendices contain the lengthier proofs of our results.

2. Monotonic redistribution rules and proportional taxation

Casajus (2014) considers redistribution in a particularly simple model of a society. For

\textsuperscript{3}Casajus (2014, Theorem 1) does not hold true for two-person societies.

\textsuperscript{4}Efficiency: redistribution has no cost. Symmetry: members of the society with the same performance obtain the same reward after redistribution. Monotonicity: whenever both the performance of a certain member of the society as well as the overall performance of the society do not decrease, then this member’s reward after redistribution should not decrease.
The properties of redistribution in the following sense. Individual performances are taxed at a certain rate and the overall tax revenue is distributed equally among the society’s members.\(^5\) The properties of redistribution rules mentioned in the introduction are formally defined as follows.

**Efficiency, E.** For all \( x \in \mathbb{R}^n \), we have \( \sum_{\ell \in \mathbb{N}_n} f_{\ell} (x) = x \).

**Symmetry, S.** For all \( x \in \mathbb{R}^n \) and \( i, j \in \mathbb{N}_n \) such that \( x_i = x_j \), we have \( f_i (x) = f_j (x) \).

**Monotonicity, M.** For all \( x, y \in \mathbb{R}^n \) and \( i \in \mathbb{N}_n \) such that \( \sum_{\ell \in \mathbb{N}_n} x_{\ell} \geq \sum_{\ell \in \mathbb{N}_n} y_{\ell} \) and \( x_i \geq y_i \), we have \( f_i (x) \geq f_i (y) \).

Cum grano salis, these properties already imply proportional taxation for societies comprising more than two members\(^6\) in the following sense. Individual performances are taxed at a certain rate and the overall tax revenue is distributed equally among the society’s members.\(^7\)

**Theorem 1 (Casajus, 2014).** Let \( n > 2 \). A redistribution rule \( f : \mathbb{R}^n \to \mathbb{R}^n \) satisfies efficiency (E), symmetry (S), and monotonicity (M) if and only if there exists some \( \tau \in [0, 1] \) such that

\[
    f_i (x) = (1 - \tau) \cdot x_i + \frac{1}{n} \cdot \sum_{\ell \in \mathbb{N}_n} \tau \cdot x_{\ell} \quad \text{for all } x \in \mathbb{R}^n \text{ and } i \in \mathbb{N}_n.
\]

The proof of this result, in particular, the proof that the tax rate does not depend on the overall performance of the society makes use of an unbounded-domain assumption, i.e., the fact that we consider arbitrary great or small (negative) individual performances. Since the interpretation of the taxation of negative performances is less convincing than for non-negative performances, one might wonder whether Theorem 1 remains true if one restricts attention to non-negative performances. That is, the domain of redistribution rules is \( \mathbb{R}_+^n \).\(^8\) Moreover, the properties are required to hold only for \( x, y \in \mathbb{R}_+^n \). We indicate this by a subscript “+” at the abbreviations of the properties.

Consider the redistribution rule \( f^\vartriangleright : \mathbb{R}_+^n \to \mathbb{R}_+^n \) given by

\[
    f_i^\vartriangleright (x) := \begin{cases} 
        0, & c = 0, \\
        (1 - \tau^\vartriangleright (c)) \cdot x_i + \frac{1}{n} \cdot \tau^\vartriangleright (c) \cdot c, & 0 < c < 1
    \end{cases}
\]

for all \( x \in \mathbb{R}_+^n \) and \( i \in \mathbb{N}_n \), \( \tau^\vartriangleright : \mathbb{R}_+^n \to [0, 1] \),

\[
    \tau^\vartriangleright (c) := \begin{cases} 
        c^{-1}, & c > 1, \\
        1, & c \leq 1
    \end{cases}
\]

for all \( c \in \mathbb{R}_+^n \).\(^9\)

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\(^5\)This setup is related to bankruptcy problems and division rules. Thomson (2003) provides a survey on these.

\(^6\)Throughout, we disregard the trivial case \( n = 1 \).

\(^7\)Casajus and Huettner (2014) obtain a similar result for one-point solutions of cooperative games with transferable utility. For bankruptcy problems and division rules, Moulin (1987, Theorem 2) establishes a related result.

\(^8\)We set \( \mathbb{R}_+ := [0, +\infty) \) and \( \mathbb{R}_{++} = (0, +\infty) \).
Note that this is quite a reasonable rule. The society wishes to guarantee its members a minimum reward of $\frac{1}{n}$ but not more. If this is not possible, the society would like to be as close as possible to this aim in the following sense. If the society is not so wealthy, $\sum_{\ell \in \mathbb{N}_n} x_{\ell} \leq 1$, redistribution is egalitarian, i.e., the tax rate is 1 and all members of the society end up with the same reward. In a more affluent society, $\sum_{\ell \in \mathbb{N}_n} x_{\ell} > 1$, redistribution is restricted to the level for $\sum_{\ell \in \mathbb{N}_n} x_{\ell} = 1$. That is, the tax rate is set to $(\sum_{\ell \in \mathbb{N}_n} x_{\ell})^{-1}$, which entails an overall tax revenue amounting to 1. It is straightforward to show that the redistribution rule $f^\circ$ satisfies the restricted versions of efficiency, symmetry, and monotonicity. Hence, the restriction of Casajus (2014, Theorem 1) to non-negative performances does not hold true.

This triggers two questions. What are the implications of the restricted properties? How can the original result be restored in the restricted setup?

The first question is answered by the next theorem. Its proof is referred to Appendix A. The restriction of Casajus’ counterexample for $n = 2$ also works on the domain of non-negative allocations.

**Theorem 2.** Let $n > 2$. A redistribution rule $f : \mathbb{R}_+^n \rightarrow \mathbb{R}^n$ satisfies efficiency ($E_+$), symmetry ($S_+$), and monotonicity ($M_+$) if and only if there is a mapping $\tau : \mathbb{R}_{++} \rightarrow [0, 1]$ with the following properties.

(i) For all $x \in \mathbb{R}_+^n$ and $i \in \mathbb{N}_n$, we have

$$f_i (x) = \begin{cases} 0, & c = 0, \\ (1 - \tau (c)) \cdot x_i + \frac{1}{n} \cdot \tau (c) \cdot c, & c > 0, \end{cases}$$

where $c = \sum_{\ell \in \mathbb{N}_n} x_{\ell}$. \hfill (4)

(ii) For all $c, d \in \mathbb{R}_{++}$ such that $d \geq c$, we have $\tau (d) \cdot d \geq \tau (c) \cdot c$.

(iii) For all $c, d \in \mathbb{R}_{++}$ such that $d \geq c$, we have

$$\frac{\tau (d) \cdot d}{n} - \frac{\tau (c) \cdot c}{n} \geq \frac{(\tau (d) - \tau (c)) \cdot c}{n}.$$
not allowed to increase more than the per-capita tax with increasing total performance. This property “protects” the strongest member of a society, i.e., a member whose performance equals overall performance, because a sole performer pays all the tax.

As an (almost) immediate consequence of Theorem 2, zero taxation has to be global, i.e., the tax rate is either zero for all positive overall performances or positive for all positive overall performances.

**Corollary 3.** Let \( n > 2 \). If a tax rate function \( \tau : \mathbb{R}^+ \to [0, 1] \) meets properties (ii) and (iii) of Theorem 2 and \( \tau(d) = 0 \) for some \( d \in \mathbb{R}^+ \), then \( \tau(c) = 0 \) for all \( c \in \mathbb{R}^+ \).

**Proof.** Let \( \tau : \mathbb{R}^+ \to [0, 1] \) meet properties (ii) and (iii) of Theorem 2. Moreover, let \( d \in \mathbb{R}^+ \) be such that \( \tau(d) = 0 \). By (ii), \( 0 \geq \tau(c) \cdot c \), i.e., \( \tau(c) = 0 \) for all \( c \in (0, d] \). By (iii), \( \tau(c) \cdot \left( \frac{c}{n} - d \right) \geq 0 \), i.e., \( \tau(c) = 0 \) for all \( c \in [d, n \cdot d] \). Hence, \( \tau(c) = 0 \) for all \( c \in [d, \infty) \). \( \square \)

In the next theorem, we characterize the tax rate functions from Theorem 2 by local conditions. Its proof is referred to Appendix B.

**Theorem 4.** Let \( n > 2 \). A mapping \( \tau : \mathbb{R}^+ \to [0, 1] \) satisfies conditions (ii) and (iii) of Theorem 2 if and only if it satisfies the following conditions.

(a) The mapping \( \tau \) is absolutely continuous on any compact subinterval of \( \mathbb{R}^+ \).

(b) If \( \tau \) is differentiable at \( c \in \mathbb{R}^+ \), then \( -\tau(c) \leq \tau'(c) \cdot c \).

(c) If \( \tau \) is differentiable at \( c \in \mathbb{R}^+ \), then \( \tau'(c) \cdot c \leq \frac{1}{n-1} \cdot \tau(c) \).

In view of Corollary 3, we may focus on tax rate functions that never assign a zero tax rate. For such functions, conditions (ii) and (iii) can be rewritten as

\[
-1 \leq \frac{\tau'(c) \cdot c}{\tau(c)} \leq \frac{1}{n-1},
\]

i.e., conditions (b) and (c) essentially represent a lower and an upper bound on the elasticity of the tax rate function. This indicates that there is rich family of tax rate functions that are compatible with the three properties of redistribution. Below, we provide a kind of pathological example that jumps back and forth between both boundaries.

Some technical remarks on Theorem 4 seem to be in order: Since continuity is a local property and since absolute continuity implies continuity, the tax rate function is continuous on its whole domain. Moreover, in the proof of the theorem, we actually show that tax rate function even is Lipschitz continuous on any subinterval that does not contain an open neighborhood of 0. However, the tax rate function neither needs to be continuously extendable to 0 nor to be absolutely continuous on \( \mathbb{R}^+ \). Let the tax rate function \( \tau^\kappa : \mathbb{R}^+ \to [0, 1] \) be given by

\[
\tau^\kappa(c) := \begin{cases}
\frac{1}{2^{(k+1) \cdot n}} \cdot c, & \frac{1}{2^{(k+1) \cdot n}} < c \leq \frac{1}{2^{(k+1) \cdot n-1}}, \\
\frac{1}{2^{k \cdot n}} \cdot c \cdot \frac{1}{n-1}, & \frac{1}{2^{(k+1) \cdot n-1}} < c \leq \frac{1}{2^{2k \cdot n}}, \\
& \text{for all } c \in \mathbb{R}^+ \text{ and } k \in \mathbb{Z}.
\end{cases}
\]
It is straightforward to show that domain and range of $\tau^\mathbb{R}$ are as in Theorem 4 as well as that $\tau^\mathbb{R}$ satisfies conditions (a), (b), and (c). In particular, the elasticities of $\tau^\mathbb{R}$ on the interior of the subdomains in (6) are $-1$ and $\frac{1}{n-1}$, respectively. Yet, in any open neighborhood of 0, $\tau^\mathbb{R}$ assumes any value between $\frac{1}{2}$ and 1 (compare Royden, 1988, Problem 12, p. 110).

We conclude this section by answering the second question. Obviously, the redistribution rule $f^\otimes$ is not invariant under upscaling performances, i.e., $f^\otimes$ fails the following property for $n > 1$.

**Homogeneity, $H_+$.** For all $x \in \mathbb{R}^n$, $i \in \mathbb{N}_n$, and $\rho \in \mathbb{R}_+$, we have $f_i (\rho \cdot x) = \rho \cdot f_i (x)$.

On the one hand, homogeneity could be interpreted as that redistribution is not affected by the currency used. Given this interpretation homogeneity is a rather natural requirement on redistribution rules. Interpreting homogeneity as scale invariance of redistribution, on the other hand, it is not that innocuous. Anyway, adding this property to the list of Casajus (2014, Theorem 1) restores its implications when restricted to non-negative performances. Note that the restriction of Casajus’ counterexample for $n = 2$ also meets homogeneity.

**Theorem 5.** Let $n > 2$. A redistribution rule $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ satisfies efficiency ($E_+$), symmetry ($S_+$), monotonicity ($M_+$), and homogeneity ($H_+$) if and only if there exists some $\tau \in [0, 1]$ such that

$$f_i (x) = (1 - \tau) \cdot x_i + \frac{1}{n} \cdot \sum_{\ell \in \mathbb{N}_n} \tau \cdot x_\ell \quad \text{for all } x \in \mathbb{R}^n_+ \text{ and } i \in \mathbb{N}_n. \quad (7)$$

**Proof.** Let $n > 2$. By Theorem 2, the redistribution rule in (7) satisfies $E_+$, $S_+$, and $M_+$. Since $\tau \in [0, 1]$ and $x \in \mathbb{R}^n_+$, it also meets $H_+$. Let $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ meet $E_+$, $S_+$, $M_+$, and $H_+$. By Theorem 2, there is a mapping $\tau : \mathbb{R}^{++} \to \mathbb{R}$ such that $f$ is as in (4). Let $c \in \mathbb{R}^{++}$ and $x, y \in \mathbb{R}^n_+$ be such that $x_\ell = y_\ell = 0$ for all $\ell \in \{1, \ldots, n - 1\}$, $x_n = 1$, and $y_n = c$, i.e., $y = c \cdot x$. Hence, we have

$$c \cdot \frac{1}{n} \cdot \tau (1) \cdot 1 \overset{(4)}{=} c \cdot f_1 (x) \overset{H_+}{=} f_1 (y) \overset{(4)}{=} \frac{1}{n} \cdot \tau (c) \cdot c$$

and therefore $\tau (c) = \tau (1)$ for all $c \in (0, \infty)$. \hfill \qed

3. **Concluding remarks**

The main insight of this paper is that monotonic redistribution of non-negative allocations breathes the spirit of Casajus’ (2014) characterization. In particular, monotonic redistribution implies that individual taxation essentially does not depend on individual performance but on overall performance. Yet, the elasticity of the tax rate with respect to overall performance is bounded above and below.

One limitation of our approach is that, for example, we do not distinguish between a low individual performance due to low potential and a low individual performance due to low effort. In the former case, the society might wish support such a member of the society via redistribution, while in the latter case it wouldn’t. Hence, it seems to desirable to study redistribution in a framework where members of the society are described by both their actual performance and their potential.
Appendix A. Proof of Theorem 2

First, the only-if part. Let $n > 2$ and let $f : \mathbb{R}^n_+ \to \mathbb{R}^n$ meet $E_+, S_+$, and $M_+$. Set $\mathcal{R}^2_+ := \{ x \in \mathbb{R}^2_+ \mid x_1 \leq x_2 \}$. By $M_+$, there are mappings $F_i : \mathcal{R}^2_+ \to \mathbb{R}$, $i \in \mathbb{N}_n$ such that

$$f_i(x) = F_i \left( x_1, \sum_{\ell \in \mathbb{N}_n} x_\ell \right) \quad \text{for all } x \in \mathbb{R}^n_+ \text{ and } i \in \mathbb{N}_n. \quad (A.1)$$

Next, we show that $F_i = F_j =: F$ for all $i, j \in \mathbb{N}_n$. Let $(a, c) \in \mathcal{R}^2_+$ and $i, j, k \in \mathbb{N}_n$, $i \neq j \neq k \neq i$. Let $y, z \in \mathbb{R}^n_+$ be given by

$$y_i = a \quad \text{and} \quad y_\ell = \frac{c - a}{n - 1} \quad \text{for all } \ell \in \mathbb{N}_n \setminus \{i\} \quad (A.2)$$

and

$$z_j = a \quad \text{and} \quad z_\ell = \frac{c - a}{n - 1} \quad \text{for all } \ell \in \mathbb{N}_n \setminus \{j\}. \quad (A.3)$$

We have

$$F_i(a, c) \overset{(A.1)}{=} f_i(y) \overset{(A.2), E_+, S_+}{=} c - (n - 1) \cdot f_k(y) \overset{(A.2), (A.3), M_+}{=} c - (n - 1) \cdot f_k(z) \overset{(A.3), E_+, S_+}{=} f_j(z) \overset{(A.1)}{=} F_j(a, c). \quad (A.5)$$

By $E_+$ and $M_+$, the mapping $F$ has the following properties.

**Efficiency, $E^*$.** For all $a \in \mathbb{R}^n_+$, we have $\sum_{\ell \in \mathbb{N}_n} F \left( a_\ell, \sum_{k \in \mathbb{N}_n} a_k \right) = \sum_{\ell \in \mathbb{N}_n} a_\ell$.

**Monotonicity, $M^*$.** For all $a, b, c, d \in \mathbb{R}_+$ such that $b \geq a$ and $d \geq c$, we have $F(b, d) \geq F(a, c)$.

For $c \in \mathbb{R}_+, c > 0$, let the mapping $\Phi^c : [0, c] \to \mathbb{R}$ be given by

$$\Phi^c(a) := F(a, c) - F(0, c) \quad \text{for all } a \in [0, c]. \quad (A.4)$$

For $a, b \in [0, c], a + b \leq c$, we have

$$F(a, c) + F(b, c) + (n - 2) \cdot F \left( \frac{c - a - b}{n - 2}, c \right) \overset{E^*}{=} F(a + b, c) + F(0, c) + (n - 2) \cdot F \left( \frac{c - a - b}{n - 2}, c \right). \quad (A.5)$$

By (A.4) and (A.5), we further obtain

$$\Phi^c(a) + \Phi^c(b) = \Phi^c(a + b). \quad (A.6)$$

This already entails

$$\Phi^c(\rho \cdot a) = \rho \cdot \Phi^c(a) \quad \text{for all } a \in [0, c] \text{ and } \rho \in \mathbb{Q}_+. \quad (A.7)$$
such that \( \rho \cdot a \in [0, c] \). By \( M^* \), \( \Phi^c \) is monotonic, i.e., \( \Phi^c(a) \geq \Phi^c(b) \) for all \( a, b \in [0, c] \) such that \( a \geq b \). Since \( \mathbb{Q}_+ \) is a dense subset of \( \mathbb{R}_+ \), (A.7) entails
\[
\Phi^c(\rho \cdot a) = \rho \cdot \Phi^c(a) \quad \text{for all } a \in [0, c] \text{ and } \rho \in \mathbb{R}_+
\] (A.8)
such that \( \rho \cdot a \in [0, c] \).

By \( E^* \), \( F(0, 0) = 0 \). For all \( c \in \mathbb{R}_+ \), set
\[
\alpha_c := \frac{F(c, c) - F(0, c)}{c}.
\] (A.9)

By (A.4), (A.8), and (A.9), we have
\[
F(a, c) = \alpha_c \cdot a + F(0, c) \quad \text{for all } (a, c) \in \mathcal{R}_+^2.
\] (A.10)

Moreover, we obtain
\[
\frac{c}{n} E^* = \frac{F\left(\frac{c}{n}, c\right)}{c} \overset{(A.10)}{=} \alpha_c \cdot \frac{c}{n} + F(0, c),
\]
i.e., \( F(0, c) = (1 - \alpha_c) \cdot \frac{c}{n} \) for all \( c \in \mathbb{R}_+ \) and therefore
\[
F(a, c) = \alpha_c \cdot a + (1 - \alpha_c) \cdot \frac{c}{n} \quad \text{for all } (a, c) \in \mathcal{R}_+^2, \ c > 0.
\] (A.11)

By (A.9) and \( M^* \), we have \( \alpha_c \geq 0 \) for all \( c \in \mathbb{R}_+ \). Moreover, we have
\[
0 \overset{E^*}{=} F(0, 0) \overset{M^*}{\leq} F(0, c) \overset{(A.11),(A.12)}{=} (1 - \alpha_c) \cdot \frac{c}{n},
\]
i.e., \( 1 \geq \alpha_c \) for all \( c \in \mathbb{R}_+ \). Hence, the range of the mapping \( \tau : \mathbb{R}_+ \rightarrow \mathbb{R} \) given by
\[
\tau(c) := 1 - \alpha_c \quad \text{for all } c \in (0, \infty)
\] (A.12)
is as in the theorem. By (A.1), (A.11), and (A.12), \( \tau \) also meets part (i).

For \( c, d \in \mathbb{R}_+ \), \( d > c \), we have
\[
\tau(c) \cdot \frac{c}{n} \overset{(A.11),(A.12)}{=} F(0, c) \overset{M^*, (A.11),(A.12)}{\leq} F(0, d) \overset{(A.11),(A.12)}{=} \tau(d) \cdot \frac{d}{n},
\]
i.e., \( \tau \) satisfies part (ii) of the theorem. Part (iii) drops from
\[
(1 - \tau(c)) \cdot c + \tau(c) \cdot \frac{c}{n} \overset{(A.11),(A.12)}{=} F(c, c) \overset{M^*, (A.11),(A.12)}{\leq} F(c, d) \overset{(A.11),(A.12)}{=} (1 - \tau(d)) \cdot c + \tau(d) \cdot \frac{d}{n}.
\]

And now, the if part. Let the mapping \( \tau : \mathbb{R}_+ \rightarrow [0, 1] \) obey (i)–(iii) and let the redistribution rule \( f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n \) be given as in (4). It is immediate that \( f \) meets \( E \) and \( S \).
Let \( x, y \in \mathbb{R}_+^n \) and \( i \in \mathbb{N}_n \) be such that \( y_i \geq x_i \) and \( d := \sum_{i \in \mathbb{N}_n} y_i \geq \sum_{i \in \mathbb{N}_n} x_i =: c \). If \( c = 0 \), then \( x_i = 0 \) and \( f_i(x) = 0 \leq f_i(y) \) by (4) and \( \tau(d) \in [0, 1] \). For \( c > 0 \), we have

\[
 f_i(y) \overset{(4)}{=} (1 - \tau(d)) \cdot y_i + \tau(d) \cdot \frac{d}{n} \geq (1 - \tau(d)) \cdot x_i + \tau(d) \cdot \frac{d}{n} = \frac{x_i}{c} \cdot \left[ (1 - \tau(c)) \cdot c + \tau(c) \cdot \frac{d}{n} \right] + \frac{c - x_i}{c} \cdot \left[ \tau(c) \cdot \frac{c}{n} \right] \overset{(ii),(iii)}{=} (1 - \tau(c)) \cdot x_i + \tau(c) \cdot \frac{c}{n} \overset{(4)}{=} f_i(x).
\]

Hence, \( f \) also meets \( \mathbf{M}_+ \). \( \square \)

**Appendix B. Proof of Theorem 4**

First, the if part. Let \( n > 2 \). Further, let the mapping \( \tau : \mathbb{R}_{++} \rightarrow [0, 1] \) satisfy (a), (b), and (c). Fix \( c, d \in \mathbb{R}_{++} \), \( d > c \). By (a) and Royden (1988, p. 110) \( \tau \) is differentiable almost everywhere on \([c, d]\), its derivative \( \tau' \) is Lebesgue integrable, and

\[
 \tau(a) = \tau(c) + \int_{c}^{a} \tau'(t) \, dt \quad \text{for all } a \in [c, d].
\]

Consider the mapping \( R : [c, d] \rightarrow \mathbb{R}_+ \) given by \( R(a) = \tau(a) \cdot a \) for all \( a \in [c, d] \). By construction, \( R \) is absolutely continuous, i.e., \( R \) is differentiable almost everywhere on \([c, d]\), its derivative \( R' \) is Lebesgue integrable, and

\[
 R(d) = R(c) + \int_{c}^{d} R'(t) \, dt. \tag{B.1}
\]

If \( \tau \) is differentiable at \( t \in [c, d] \), then \( R \) is also differentiable at \( t \) and \( R'(t) = \tau'(t) \cdot t + \tau(t) \). By (b), \( R'(t) \geq 0 \). Hence, (B.1) implies \( \tau(d) \cdot d = R(d) \geq R(c) = \tau(c) \cdot c \). Since \( c \) and \( d \) were chosen arbitrarily, \( \tau \) satisfies condition (ii) of Theorem 2.

Consider the mapping \( S : [c, d] \rightarrow \mathbb{R}_+ \) given by \( S(a) = \frac{1}{n} \cdot \tau(a) \cdot a - \tau(a) \cdot c \) for all \( a \in [c, d] \). If \( \tau(d) \leq \tau(c) \), then (ii) implies \( S(d) \geq S(c) \). Suppose \( \tau(d) > \tau(c) \). By (a), \( \tau \) is continuous. Hence, there is some \( b \in (c, d] \) such that

\[
 \tau(b) = \tau(d) \quad \text{and} \quad \tau(a) \leq \tau(d) \quad \text{for all } a \in [c, b]. \tag{B.2}
\]

Thus,

\[
 S(d) = \frac{1}{n} \cdot \tau(d) \cdot d - \tau(d) \cdot c \geq \frac{1}{n} \cdot \tau(b) \cdot b - \tau(b) \cdot c = S(b). \tag{B.3}
\]
By construction, $S$ is absolutely continuous, i.e., $S$ is differentiable almost everywhere on $[c, d]$, the derivative $S'$ is Lebesgue integrable, and

$$S(a) = S(c) + \int_c^a S'(t) \, dt \quad \text{for all } a \in [c, d]. \quad (B.4)$$

If $\tau$ is differentiable at $t \in [c, d]$, then $S$ is also differentiable at $t$ and

$$S'(t) = \frac{1}{n} \cdot \tau'(t) \cdot t + \frac{1}{n} \cdot \tau(t) - \tau'(t) \cdot c \quad (c)$$

$$\geq \frac{1}{n} \cdot \tau'(t) \cdot t + \frac{1}{n} \cdot (n-1) \cdot \tau'(t) \cdot d - \tau'(t) \cdot c$$

$$= \tau'(t) \cdot (t - c). \quad (B.5)$$

Hence, we have

$$S(b) \overset{(B.4),(B.5)}{\geq} S(c) + \int_c^b \tau'(t) \cdot (t - c) \, dt$$

$$= S(c) + \int_c^b \tau'(t) \cdot t + \tau(t) \, dt - \int_c^b \tau'(t) \cdot c \, dt$$

$$= S(c) + \tau(b) \cdot (b - c) - \int_c^b \tau(t) \, dt$$

$$= S(c) + \int_c^b \tau(b) - \tau(t) \, dt \quad (B.2)$$

$$\overset{(B.3),(B.6)}{\geq} S(c). \quad (B.6)$$

Finally, we obtain

$$\frac{1}{n} \cdot \tau(d) \cdot d - \tau(d) \cdot c = S(d) \overset{(B.3),(B.6)}{\geq} S(c) = \frac{1}{n} \cdot \tau(c) \cdot c - \tau(c) \cdot c.$$

Since $c$ and $d$ were chosen arbitrarily, $\tau$ satisfies condition (iii) of Theorem 2.

And now, the only-if part. Let the mapping $\tau : \mathbb{R}^+_+ \rightarrow [0, 1]$ satisfy conditions (ii) and (iii) of Theorem 2. Fix $c, d \in \mathbb{R}^+_+, d > c$. For $a, b \in [c, d], b > a$, we have

$$\tau(b) - \tau(a) \overset{(ii)}{\geq} - \frac{\tau(b)}{a} \cdot (b - a) \quad \overset{\tau(b) \in [0,1], b > a \geq c}{\geq} - \frac{1}{c} \cdot (b - a) \quad (B.7)$$

and

$$\frac{1}{n - 1} \cdot \frac{1}{c} \cdot (b - a) \overset{\tau(b) \in [0,1], b > a \geq c}{\geq} \frac{1}{n - 1} \cdot \frac{\tau(b)}{a} \cdot (b - a) \overset{(iii)}{\geq} \tau(b) - \tau(a). \quad (B.8)$$

Hence,

$$|\tau(b) - \tau(a)| \leq \frac{1}{c} \cdot |b - a|.$$
Since $a$ and $b$ were chosen arbitrarily, $\tau$ is Lipschitz continuous on $[c, d]$. By Royden (1988, Problem 20 (a), p. 112), $\tau$ is absolutely continuous on $[c, d]$.

Let $\tau$ be differentiable at $a \in \mathbb{R}_{++}$. By (B.7), we have

$$\frac{\tau(b) - \tau(a)}{b - a} \cdot a \geq -\tau(b) \quad \text{for all } b \in (a, \infty).$$

Taking the limit as $b$ approaches $a$ gives $\tau'(a) \cdot a \geq -\tau(a)$, where we make use of the fact an absolutely continuous mapping is continuous and that continuity is a local property. Hence, $\tau$ satisfies (b). By (B.8), we have

$$\frac{1}{n - 1} \cdot \tau(b) \geq \frac{\tau(b) - \tau(a)}{b - a} \cdot a \quad \text{for all } b \in (a, \infty).$$

Taking the limit as $b$ approaches $a$ gives $\frac{1}{n - 1} \cdot \tau(a) \geq \tau'(a) \cdot a$. Hence, $\tau$ satisfies (c). \qed

References


