Potential, Value, and the Multilinear Extension

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Abstract:
We provide new formulae for the potential of the Shapley value that use the multilinear extension of coalitional games with transferable utility.
Potential, value, and the multilinear extension✩

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Abstract

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1. Introduction

A cooperative game with transferable utility (TU game) consists of a finite set of players and a coalition function that assigns a worth to any subset of the player set. Assuming that all players cooperate, the question arises how to distribute the overall worth generated by them. This question is answered by solution concepts. The Shapley value (Shapley, 1953) probably is the most eminent one-point solution concept for cooperative games with transferable utility.

Hart and Mas-Colell (1989) suggest an indirect characterization of the Shapley value by a potential (function). A potential is a function that assigns a real number to any TU game with the following properties. (i) The potential of the game on the empty player set is zero. (ii) The marginal contribution of a player in a game with respect to the potential is the difference between the potential of this game and the potential of the game without this player. For any game, these marginal contributions sum up to the worth generated by the game’s grand coalition. Hart and Mas-Colell (1989, Theorem A) show that there exists a unique potential and that the marginal contributions coincide with the Shapley payoffs.\textsuperscript{1}

Hart and Mas-Colell (1989, Proposition 2.4) also show that the per-capita potential of a game is the expected per-capita worth of a standard random coalition. They conclude that this fact indicates that

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1 Calvo and Santos (1997) and Ortmann (1998) generalize the notion of a potential.
“... the potential provides the most natural one-number summary of a game.”

—Hart and Mas-Colell (1989, p. 592)

Casajus (2014) supports this view by showing that the potential is the expected worth generated by some natural random partition of the player set.

Owen (1972) introduces the multilinear extension of a TU game. The domain of this extension is the standard cube, representing the players’ probabilities of participating in the generation of worth. He obtains the Shapley value as the integral of partial derivatives of the multilinear extension alongside the diagonal of the standard cube. This formula is particularly useful when computing the Shapley value for large (voting) games.

In this paper, we suggest two new formulae for the potential that use the multilinear extension. While the above mentioned formula for the Shapley value employs the partial derivatives, we employ the total differential alongside the diagonal of the standard cube. Moreover, we make use of a non-uniform probability distribution on the unit interval. Second, we show that the potential of a game is the double integral of the total differential of the multilinear extension alongside the diagonal of the standard cube.

This note is organized as follows. Basic definitions and notation are given in the second section. The third section contains our results. Some remarks conclude this note.

2. Basic definitions and notation

Let \( N \) be a non-empty and finite set. For \( S, T \subseteq N, s, t, \) and \( n \) denote their cardinalities, respectively. A (TU) game on \( N \) is given by a coalition function \( v \in \mathcal{V} := \{ f : 2^N \to \mathbb{R} \mid f(\emptyset) = 0 \} \). For \( v, w \in \mathcal{V} \) and \( \rho \in \mathbb{R}, \) the games \( v + w \in \mathcal{V} \) and \( \rho \cdot v \in \mathcal{V} \) are given by \( (v + w)(S) = v(S) + w(S) \) and \( (\rho \cdot v)(S) = \rho \cdot v(S) \) for all \( S \subseteq N. \)

For \( T \subseteq N, T \neq \emptyset, \) the unanimity game \( u_T \in \mathcal{V} \) is given by \( u_T(S) = 1 \) if \( T \subseteq S \) and \( u_T(S) = 0 \) if \( T \not\subseteq S \) for all \( S \subseteq N. \) For \( v \in \mathcal{V}, \) we have

\[
v = \sum_{T \subseteq N : T \neq \emptyset} \lambda_T(v) \cdot u_T, \quad \lambda_T(v) := \sum_{S \subseteq T : S \neq \emptyset} (-1)^{|T|-|S|} \cdot v(S). \tag{1}
\]

The multilinear extension \( \bar{v} : [0, 1]^N \to \mathbb{R} \) of \( v \in \mathcal{V} \) is given by

\[
\bar{v}(x) = \sum_{S \subseteq N : S \neq \emptyset} \prod_i x_i \prod_{i \in N \setminus S} (1 - x_i) \cdot v(S) \quad \text{for all } x \in [0, 1]^N. \tag{2}
\]

A solution is an operator \( \varphi \) that assigns a payoff vector \( \varphi(v) \in \mathbb{R}^N \) to any \( v \in \mathcal{V}. \) The Shapley value (Shapley, 1953) is given by

\[
\text{Sh}_i(v) = \sum_{T \subseteq N : T \ni i} \frac{\lambda_T(v)}{t} \quad \text{for all } v \in \mathcal{V} \text{ and } i \in N. \tag{3}
\]

The potential (Hart and Mas-Colell, 1989) is given by

\[
\text{Pot}(v) := \sum_{T \subseteq N : T \neq \emptyset} \frac{\lambda_T(v)}{t} \quad \text{for all } v \in \mathcal{V}. \tag{4}
\]
3. The potential via the multilinear extension

Owen (1972) derives the following formula for the Shapley value,

\[ \text{Sh}_i(v) = \int_0^1 \frac{\partial \bar{v}}{\partial x_i}(t, \ldots, t) \, dt \quad \text{for all } v \in \mathcal{V} \text{ and } i \in N. \]

In this section, we show that the potential can be obtained by similar expressions. Instead of the partial derivatives, we employ the total differential alongside the diagonal of the standard cube. Moreover, we make use of a non-uniform probability distribution on the unit interval.

**Theorem 1.** For all \( v \in \mathcal{V} \), we have

\[ \text{Pot}(v) = \int_0^1 -\ln \theta \cdot \frac{d\bar{u}_T(\xi, \ldots, \xi)}{d\xi}(\theta) \, d\theta. \]  

**Proof.** Since the multi-linear extension and its partial derivatives are linear in \( v \), and the integral is linear, it suffices to show the claim for all \( u_T, T \subseteq N, T \neq \emptyset \).

Fix \( T \subseteq N, T \neq \emptyset \). By (2), we have \( \bar{u}_T(x) = \prod_{\ell \in T} x_\ell \) and therefore

\[ \frac{d\bar{u}_T(\xi, \ldots, \xi)}{d\xi}(\theta) = \sum_{i \in N} \frac{\partial \bar{u}_T(\theta, \ldots, \theta)}{\partial x_i}(t \cdot \theta^{t-1}) \quad \text{for all } \theta \in [0, 1]. \]  

Moreover, we have

\[ \frac{d}{d\theta} \left( \frac{1}{t} \cdot \theta^t \cdot (1 - \theta \cdot \ln \theta) \right) = -\ln \theta \cdot t \cdot \theta^{t-1} \]  

and

\[ \lim_{\theta \to 0} \frac{1}{t} \cdot \theta^t \cdot (1 - \theta \cdot \ln \theta) = \lim_{\theta \to 0} \frac{-\theta \cdot \ln \theta}{\theta^{t-1}} = \lim_{\theta \to 0} \frac{-\theta}{t \cdot \theta^{t-1}} = 0, \]

where the first equation drops from l’Hôpital’s rule. Thus, we obtain

\[ \int_0^1 -\ln \theta \cdot \frac{d\bar{u}_T(\xi, \ldots, \xi)}{d\xi}(\theta) \, d\theta \overset{(6)}{=} \int_0^1 -\ln \theta \cdot t \cdot \theta^{t-1} \, d\theta \]

\[ \overset{(7)}{=} \frac{1}{t} \cdot \theta^t \cdot (1 - \theta \cdot \ln \theta) \bigg|_0^1 = \frac{1}{t} \overset{(1), (4)}{=} \text{Pot}(u_T). \]

This concludes the proof. \( \square \)

**Remark 2.** By (6), we have \( \int_0^1 \frac{d\bar{u}_T(\xi, \ldots, \xi)}{d\xi}(\theta) \, d\theta = 1 = u_T(N) \) for all \( T \subseteq N, T \neq \emptyset \). The linearity of the multi-linear extension and its total differential in \( v \), and the linearity of the integral thus entail

\[ \int_0^1 \frac{d\bar{v}(\xi, \ldots, \xi)}{d\xi}(\theta) \, d\theta = v(N) \quad \text{for all } v \in \mathcal{V}. \]  

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Compare this with (5). While all $\theta$ are equally likely in (9), lower probabilities of cooperation $\theta$ are more likely in (5). This corresponds, for example, to the fact that in the formula for the potential (4) the coefficients $\lambda_T(v)$ are divided by $t$, whereas we have $v(N) = \sum_{T \subseteq N : T \neq \emptyset} \lambda_T(v)$.

The next theorem establishes that the potential of a game is the double integral of the total differential of the multilinear extension alongside the diagonal of the standard cube.

**Theorem 3.** For all $v \in \mathcal{V}$, we have
\[
\text{Pot}(v) = \int_0^1 \int_0^1 \frac{d\bar{v}(\xi, \ldots, \xi)}{d\xi}(\theta_1 \cdot \theta_2) \, d\theta_1 d\theta_2.
\] (10)

**Proof.** Let $X_1$ and $X_2$ be independent random variables on $[0, 1]$, each with uniform distribution. Then, their product is a probability distribution on $[0, 1]$ given by the probability function $F$,
\[
F(a) = a \cdot (1 - \ln a) \quad \text{for all } a \in [0, 1],
\]
which can be seen from
\[
F(a) = \int_0^1 \min \left(1, \frac{a}{\theta_1} \right) \, d\theta_1 = \int_0^a 1 \, d\theta_1 + \int_a^1 \frac{a}{\theta_1} \, d\theta_1 = a - a \cdot \ln a.
\]
The density function $d$ of $F$ is given by
\[
d(a) = -\ln a \quad \text{for all } a \in (0, 1].
\] (11)
Hence, we obtain
\[
\int_0^1 \int_0^1 \frac{d\bar{v}(\xi, \ldots, \xi)}{d\xi}(\theta_1 \cdot \theta_2) \, d\theta_1 d\theta_2 \overset{(11)}{=} \int_0^1 -\ln \theta \cdot \frac{d\bar{v}(\xi, \ldots, \xi)}{d\xi}(\theta) \, d\theta = \text{Pot}(v),
\]
where the last equation drops from Theorem 1. \qed

4. **Concluding remarks**

In this paper, we provide new formulae for the potential of the Shapley value. Along the lines of Owen (1972) and Leech (2003), one can use these formulae in order to approximate the potential of large voting games. This is particularly interesting in view of Casajus’ (2014) proposal to use the potential as a measure of power concentration in voting games.

**References**


