Decomposition of Solutions and the Shapley Value

André Casajus\textsuperscript{a}, Frank Huettner\textsuperscript{b}

\textsuperscript{a} Prof. Dr. André Casajus is a Research Professor at the Chair of Economics and Information Systems at HHL Leipzig Graduate School of Management, Leipzig, Germany
Email: andre.casajus@hhl.de

\textsuperscript{b} Dr. Frank Huettner is a postdoctoral researcher in management science at ESMT European School of Management and Technology, Berlin, Germany

Abstract:
We suggest a foundation of the Shapley value via the decomposition of solutions for cooperative games with transferable utility. A decomposer of a solution is another solution that splits the former into a direct part and an indirect part. While the direct part (the decomposer) measures a player’s contribution in a game as such, the indirect part indicates how she affects the other players’ direct contributions by leaving the game. The Shapley value turns out to be unique decomposable decomposer of the naïve solution, where the naïve solution assigns to any player the difference between the worth of the grand coalition and its worth after this player left the game.
Decomposition of solutions and the Shapley value✩

André Casajusa*, Frank Huettnerb

aHHL Leipzig Graduate School of Management, Jahnallee 59, 04109 Leipzig, Germany
bESMT European School of Management and Technology, Schlossplatz 1, 10178 Berlin, Germany

Abstract

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Keywords: Decomposition, Shapley value, potential, consistency, higher-order contributions, balanced contributions

2010 MSC: 91A12 JEL: C71, D60

1. Introduction

The Shapley value (Shapley, 1953) probably is the most eminent one-point solution concept for cooperative games with transferable utility (TU games). Besides its original axiomatic foundation by Shapley himself, alternative foundations of different types have been suggested later on. Important direct axiomatic characterizations are due to Myerson (1980) and Young (1985). Hart and Mas-Colell (1989) suggest an indirect characterization as marginal contributions of a potential (function).1 Roth (1977) shows that the Shapley value can be understood as a von Neumann-Morgenstern utility. As a contribution to the Nash program, which aims at building bridges between cooperative and non-cooperative game theory, Pérez-Castrillo and Wettstein (2001) implement the Shapley value as the outcomes

✩We are grateful to the participants of several seminars, workshops, and conferences for helpful comments on our paper, particularly, to Sergiu Hart and Annick Laruelle. Financial support from the Deutsche Forschungsgemeinschaft (DFG) for André Casajus (grant CA 266/4-1) and Frank Huettner (grant HU 2205/3-1) is gratefully acknowledged.

*a Corresponding author.
Email addresses: mail@casajus.de (André Casajus), mail@frankhuettner.de (Frank Huettner)
URL: www.casajus.de (André Casajus), www.frankhuettner.de (Frank Huettner)

1Calvo and Santos (1997) and Ortmann (1998) generalize the notion of a potential.
of the sub-game perfect equilibria of a combined bidding and proposing mechanism, which is modeled by a non-cooperative extensive form game.\footnote{Ju and Wettstein (2009) suggest a class of bidding mechanisms that implement several solution concepts for TU games.}

Among the solution concepts for TU games, the Shapley value can be viewed as the measure of the players’ own productivity in a game. This view is strongly supported by Young’s (1985) characterization by three properties: efficiency, strong monotonicity, and symmetry. Efficiency says that the worth generated by the grand coalition is distributed among the players. Strong monotonicity requires a player’s payoff to increase weakly whenever her productivity, measured by her marginal contributions to coalitions of the other players, weakly increases. Symmetry ensures that equally productive players obtain the same payoff.

A naïve way to measure a particular player’s productivity within a game is to look at the difference between worth generated by all players and the worth generated by all players but this particular player. This naïve solution, however, is problematic for (at least) two reasons. First, in general, the naïve payoffs do not sum up to the worth generated by the grand coalition. Hart and Mas-Colell (1989) use this fact as a motivation of the potential approach to the Shapley value. Second and not less important, one could argue that every player’s presence is necessary for generating the naïve payoff of another player.

In order to tackle the second problem of the naïve solution mentioned above, we suggest the decomposition of solutions. A solution $\psi$ decomposes a solution $\varphi$ if it splits $\varphi$ into direct and indirect contributions in the following sense. A particular player’s payoff for $\varphi$ is the sum of her payoff for $\psi$ (direct contribution) and what the other players gain or lose under $\psi$ when this particular player leaves the game (indirect contribution). That is, the indirect contributions reflect what a player contributes to the other players’ direct contributions. We say that a solution is decomposable if there exists a decomposer, i.e., a solution that decomposes it.

As our new foundation of the Shapley value, we show that the Shapley value is the unique decomposable decomposer of the naïve solution (Theorem 3). We further show that decomposability is equivalent to a number of other properties of solutions: balanced contributions (Myerson, 1980), path independence (Hart and Mas-Colell, 1989), consistency (Calvo and Santos, 1997), and admittance of a potential (Calvo and Santos, 1997; Ortmann, 1998) (Theorem 4).

We further establish that every decomposable solution has a unique decomposer that is decomposable itself. This entails that any decomposable solution admits a resolution, i.e., a sequence of solutions starting with the former solution and in which its members are decomposed by their successors (Theorem 7). Resolutions allow to capture higher-order contributions, where for instance the third-order contribution captures what player $i$ contributes to player $j$’s contribution to player $k$’s payoff. We explore the structure of higher-order decomposers using higher-order contributions (Theorem 13).

The remainder of this paper is organized as follows. In Section 2, we give basic definitions and notation. In the third section, we formalize and study the notions of decomposition and
decomposability outlined above and present our new foundation for the Shapley value. The fourth section investigates the notion of decomposability. The fifth section relates higher-order decompositions to higher-order contributions. Some remarks conclude the paper. The appendix contains all proofs.

2. Basic definitions and notation

A (TU) game on a finite player set \( N \) is given by a coalition function \( v : 2^N \to \mathbb{R} \), \( v(\emptyset) = 0 \). The set of all games on \( N \) is denoted by \( \mathbb{V}(N) \). Let \( \mathcal{N} \) denote the set of all finite player sets.\(^3\) The cardinalities of \( S, T, N, M \in \mathcal{N} \) are denoted by \( s, t, n, \) and \( m \), respectively.

Harsanyi (1959) highlights the meaning of dividends \( \lambda_T(v) \) that measure the genuine contribution of a coalition \( T \subseteq N \) to the worth generated by the grand coalition in a game \( v \in \mathbb{V}(N) \). They are given recursively by \( \lambda_T(v) = v(T) - \sum_{S \subseteq T} \lambda_S(v) \), i.e., \( \lambda_T(v) \) represents the part of the worth \( v(T) \) that is not already generated by a proper subcoalition of \( T \). Interestingly, these dividends coincide with the coefficients of the linear representation of games by unanimity games. For \( T \subseteq N, T \neq \emptyset \), the unanimity game \( u_T \in \mathbb{V}(N) \) is given by \( u_T(S) = 1 \) if \( T \subseteq S \) and \( u_T(S) = 0 \) if \( T \not\subseteq S \) for all \( S \subseteq N \). The representation of \( v \in \mathbb{V}(N) \) is given by\(^4\)

\[
v = \sum_{T \subseteq N: T \neq \emptyset} \lambda_T(v) \cdot u_T. \tag{1}\]

A solution/value is an operator \( \varphi \) that assigns a payoff vector \( \varphi(v) \in \mathbb{R}^N \) to any \( v \in \mathbb{V}(N), N \in \mathcal{N} \). The Shapley value (Shapley, 1953) distributes the dividends \( \lambda_T(v) \) equally among the players in \( T \), i.e.,

\[
Sh_i(v) := \sum_{T \subseteq N: i \in T} \frac{\lambda_T(v)}{t} \tag{2}\]

for all \( N \in \mathcal{N}, v \in \mathbb{V}(N), \) and \( i \in N \). A solution is efficient if \( \sum_{i \in N} \varphi_i(v) = v(N) \) for all \( N \in \mathcal{N} \) and \( v \in \mathbb{V}(N) \).

In this paper, we consider situations where some players leave the game. For \( v \in \mathbb{V}(N) \) and \( M \subseteq N \), the restriction of \( v \) to \( M \) is denoted by \( v|M \in \mathbb{V}(M) \) and is given by \( v|M = v(S) \) for all \( S \subseteq M \); for \( v \in \mathbb{V}(N) \) and \( M \subseteq N \), the game without the players in \( M \), \( v^{-M} \in \mathbb{V}(N \setminus M) \), is given by \( v^{-M} = v|_{N \setminus M} \). Instead of \( v^{-i} \), we write \( v^{-i} \).

3. Decomposing the naïve solution yields the Shapley value

The naïve solution, \( N_t \), is given by

\[
N_t(v) := v(N) - v^{-i}(N \setminus \{i\}) \tag{3}\]

---

\(^3\)We assume that the player sets are subsets of some given countably infinite set \( \Omega \), the universe of players; \( \mathcal{N} \) denotes the set of all finite subsets of \( \Omega \).

\(^4\)For \( v, w \in \mathbb{V}(N) \) and \( \alpha \in \mathbb{R} \), the games \( v + w \in \mathbb{V}(N) \) and \( \alpha \cdot v \in \mathbb{V}(N) \) are given by \( (v + w)(S) = v(S) + w(S) \) and \( (\alpha \cdot v)(S) = \alpha \cdot v(S) \) for all \( S \subseteq N \).
for all $N \in \mathcal{N}$, $v \in V(N)$, and $i \in N$. It reflects a player $i$’s productivity in a naïve fashion by asking what happens to the worth generated by the grand coalition when $i$ leaves the game.

This naïve solution, however, is problematic for (at least) two reasons. First, in general, the naïve payoffs do not sum up to the worth generated by the grand coalition. Second and possibly more important, the difference $v(N) - v^{-i}(N \setminus \{i\})$ cannot not only be attributed to $i$ but also requires the cooperation of players from $N \setminus \{i\}$. Consequently, we are interested in a solution that distributes $v(N) - v^{-i}(N \setminus \{i\})$ among all players,$$
abla(N) = \sum_{\ell \in N} \left[ \nabla_{\ell}(v) - \nabla_{\ell}(v^{-i}) \right]$$where $\nabla_{i}(v^{-i}) = 0$. In other words, we are interested in a solution that decomposes the naïve solution in the above sense, i.e.,$$\nabla_i(v) = \nabla_i(v) + \sum_{\ell \in N \setminus \{i\}} \left[ \nabla_{\ell}(v) - \nabla_{\ell}(v^{-i}) \right]$$for all $N \in \mathcal{N}$, $v \in V(N)$, and $i \in N$. This motivates the following definition.

**Definition 1.** A solution $\psi$ is a **decomposer** of the solution $\varphi$ if$$\varphi_i(v) = \psi_i(v) + \sum_{\ell \in N \setminus \{i\}} \left[ \psi_{\ell}(v) - \psi_{\ell}(v^{-i}) \right] \tag{4}$$for all $N \in \mathcal{N}$, $v \in V(N)$, and $i \in N$.

A decomposer $\psi$ dissects player $i$’s payoff according to the decomposed solution $\varphi$ into a direct part $\psi_i(v)$ and an indirect part $\sum_{\ell \in N \setminus \{i\}} \left[ \psi_{\ell}(v) - \psi_{\ell}(v^{-i}) \right]$. The indirect part indicates how much a player contributes to the other players’ direct part. This suggests that the decomposition of a decomposer allows to capture higher-order contributions. Indeed, if $\chi$ is the decomposer of $\psi$ and $\psi$ is the decomposer of $\varphi$, then substitution of $\psi_i(v)$ by $\chi_i(v) + \sum_{\ell \in N \setminus \{i\}} \left[ \chi_{\ell}(v) - \chi_{\ell}(v^{-i}) \right]$ in (4) yields

$$\varphi_i(v) = \chi_i(v) + 3 \cdot \sum_{\ell \in N \setminus \{i\}} \left[ \chi_{\ell}(v) - \chi_{\ell}(v^{-i}) \right]$$

$$+ \sum_{j \in N \setminus \{i\}} \sum_{k \in N \setminus \{i,j\}} \left[ \chi_k(v) - \chi_k(v^{-j}) \right] - \chi_k(v^{-i}) - \chi_k(v^{-\{j,i\}}),$$

where player $i$ makes a contribution to player $j$’s contribution to player $k$’s payoff according to $\chi$,$$
abla_k(v) - \chi_k(v^{-j}) - \chi_k(v^{-\{j,i\}}). \tag{5}$$

In order to analyze the relation between higher-order contributions and decomposers, we need to study the decomposition of decomposers. At this point, the natural question arises whether a solution is decomposable.
Definition 2. A solution $\varphi$ is called decomposable if there exists a decomposer $\psi$ of $\varphi$.

Returning to our initial problem, we now consider decomposable decompositions of the naïve solution. As our first main result, we obtain the following foundation of the Shapley value.

Theorem 3. The Shapley value is the unique decomposable decomposer of the naïve solution.

This result is driven by the fact that the efficient solutions are the decomposers of the naive solution. Among these, the Shapley value can be viewed as the natural decomposition of the naive solution. This motivates a closer look at decomposability. We clarify its nature in the next section.

At first glance, Theorem 3 seems to imply that the Shapley value is determined by the “last” marginal contributions. A deeper look, however, reveals that this, of course, is not the case. By considering the effects of players’ leaving the game on other players’ payoffs, we relate the payoff for a game to payoffs in games with smaller player sets. Since solutions apply to games with any number of players, in the end, all marginal contributions are taken into account.

4. Decomposability

The notion of decomposability is equivalent to a number of well-known properties.

Theorem 4. The following properties of a solution $\varphi$ are equivalent:

(i) Decomposability: There exists a solution $\psi$ such that

$$\varphi_i(v) = \psi_i(v) + \sum_{\ell \in N \setminus \{i\}} [\psi_\ell(v) - \psi_\ell(v^{-i})]$$

for all $N \in \mathcal{N}$, $v \in \mathcal{V}(N)$, and $i \in N$.

(ii) Balanced contributions (Myerson, 1980): For all $N \in \mathcal{N}$, $v \in \mathcal{V}(N)$, and $i, j \in N$. we have

$$\varphi_i(v) - \varphi_i(v^{-j}) = \varphi_j(v) - \varphi_j(v^{-i}).$$

(iii) Path independence (Hart and Mas-Colell, 1989): For all $N \in \mathcal{N}$, $v \in \mathcal{V}(N)$, and all bijections $\rho, \rho' : N \to \{1, \ldots, n\}$, we have

$$\sum_{i \in N} \varphi_i(v^{-S_i(\rho)}) = \sum_{i \in N} \varphi_i(v^{-S_i(\rho')}),$$

where $S_i(\rho) := \{\ell \in N \mid \rho(\ell) > \rho(i)\}$. 

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Admittance of a potential (Calvo and Santos, 1997; Ortmann, 1998): There exists a mapping \( P : \bigcup_{N \in \mathcal{N}} \mathcal{V}(N) \to \mathbb{R} \) such that

\[
\varphi_i(v) = P(v) - P(v^{-i})
\]

for all \( N \in \mathcal{N}, v \in \mathcal{V}(N) \), and \( i \in N \).

Consistency (Calvo and Santos, 1997): For all \( N \in \mathcal{N} \) and \( v \in \mathcal{V}(N) \), we have \( \varphi(v) = \text{Sh}(v^\varphi) \), where \( v^\varphi \in \mathcal{V}(N) \) is given by

\[
v^\varphi(S) = \sum_{t \in S} \varphi_t(v|_S) \quad \text{for all } S \subseteq N.
\]

Balanced contributions states that player \( j \) suffers/gains from the removal of player \( i \) by the same amount as the other way around. Path independence can be interpreted as follows. Sequentially buying players’ out of the game by paying them their payoffs according to \( \varphi \), costs the same independent of the order in which the players are bought out. A solution admits a potential if it is the differential of some potential function on the domain of all games. A solution \( \varphi \) is consistent according to the upper definition if it can be obtained as the Shapley value of an auxiliary game. In the auxiliary game, the worth generated by a coalition equals the sum of its players’ payoffs according to \( \varphi \) in the game restricted to this coalition.

The upper theorem is driven by the fact that a solution \( \psi \) is a decomposer of the solution \( \varphi \) if and only if

\[
P^\varphi(v) = \sum_{i \in N} \psi_i(v)
\]

for all \( N \in \mathcal{N}, v \in \mathcal{V}(N) \), and \( i \in N \), where \( P^\varphi \) denotes the zero-normalized potential of \( \varphi \). This further implies that a decomposer \( \psi \) of a solution \( \varphi \) is not unique. In fact, any solution \( \psi' \) such that \( \sum_{i \in N} \psi'_i(v) = \sum_{i \in N} \psi_i(v) \) for all \( N \in \mathcal{N} \) and \( v \in \mathcal{V}(N) \) also decomposes \( \varphi \). As is indicated by Theorem 3, however, there is only one decomposer of \( \varphi \) that is decomposable itself. This is further elaborated on in the next section.

5. Higher-order decompositions

In this section, we study decompositions of decomposers. First, we provide a result that clarifies the uniqueness of decomposable decomposers.

Proposition 5. If a solution \( \varphi \) is decomposable, then there exists a unique decomposable decomposer \( \psi \) of \( \varphi \). It is given by

\[
\psi_i(v) = \sum_{T \in \mathcal{N} : i \in T} \frac{\lambda_T(v^\varphi)}{t^2}
\]

for all \( k \in \mathbb{N}, N \in \mathcal{N}, v \in \mathcal{V}(N) \), and \( i \in N \), where \( v^\varphi \) is given by (6).

\[\text{A potential } P \text{ is zero-normalized if } P(\emptyset) = 0 \text{ for } \emptyset \in \mathcal{V}(\emptyset).\]
This gives rise to the following definition.

**Definition 6.** A resolution of a solution \( \varphi \) is a sequence \( (\varphi^{(k)})_{k \in \mathbb{N}} \) of solutions such that \( \varphi^{(0)} = \varphi \) and \( \varphi^{(k+1)} \) is a decomposer \( \varphi^{(k)} \) for all \( k \in \mathbb{N} \). If a resolution exists for a solution, then the latter is called **resolvable**.

As an immediate consequence of Proposition 5, resolvability is equivalent to decomposability. Moreover, a resolution, if it exists, is unique. Therefore, the solution \( \varphi^{(k)} \) is called the \( k \)-th decomposer of \( \varphi \). The next result specifies that the formulae for the higher-order decomposers are similar to (7).

**Theorem 7.** If a solution \( \varphi \) is decomposable, then there exists a unique resolution \( (\varphi^{(k)})_{k \in \mathbb{N}} \) of \( \varphi \). It is given by

\[
\varphi^{(k)}_i (v) = \sum_{T \in \mathbb{N} : i \in T} \frac{\lambda_T (v^{\varphi})}{t^{k+1}}
\]

for all \( k \in \mathbb{N} \), \( N \in \mathcal{N} \), \( v \in \mathcal{V} (N) \), and \( i \in N \).

Note that \( \varphi^{(0)} = \varphi \), which follows from the consistency of \( \varphi \) (Theorem 4) and (2). The theorem has two implications. First, for given \( k > 0 \), the per capita dividends of greater coalitions have a lower influence on the payoff than those of small coalitions. This reflects that dividends of greater coalitions are more “endangered” by players leaving the game. Second, with increasing order of decomposition, the per capita dividends of non-singleton coalitions have a lower influence on the payoffs. As an explanation we suggest that higher-order decomposers sum up to the original value taking into account increasingly more indirect effects. The extreme case is given in the next corollary.

**Corollary 8.** For all \( N \in \mathcal{N} \), \( v \in \mathcal{V} (N) \), and any decomposable solution \( \varphi \), we have

\[
\lim_{k \to \infty} \varphi^{(k)}_i (v) = \varphi_i (v |_{\{i\}}).
\]

Since \( v^{Sh} = v \), we obtain the following corollary that provides the resolution of the Shapley value and relates it to the resolution of any decomposable \( \varphi \).

**Corollary 9.** For all \( k \in \mathbb{N} \), \( N \in \mathcal{N} \), \( v \in \mathcal{V} (N) \), and any decomposable solution \( \varphi \), we have

(i) \( Sh^{(k)} (v) = \sum_{T \in \mathbb{N} : i \in T} \frac{\lambda_T (v)}{t^{k+1}} \),

(ii) \( \varphi^{(k)} (v) = Sh^{(k)} (v^{\varphi}) \).

Next, we will investigate the concrete relationship between contributions of higher order and higher-order decompositions. To this end, we define higher-order contributions as follows. Let the players \( i \), \( j \), and \( k \) be pairwise different. The contribution of \( j \) to \( k \)'s payoff under the solution \( \varphi \) in the games \( v \) and \( v^{-i} \) is given by

\[
D_{(k,j)} \varphi (v) = \varphi_k (v) - \varphi_k (v^{-j})
\]
and
\[ D_{(k,j)} (v^{-i}) = [\varphi_k (v^{-i}) - \varphi_k (v^{-j,i})] . \]

Further, the contribution of \( i \) to \( j \)'s contribution to \( k \) is given by
\[ D_{(k,j,i)} (v) = [\varphi_k (v) - \varphi_k (v^{-i})] - [\varphi_k (v^{-i}) - \varphi_k (v^{-j,i})] = D_{(k,j)} (v) - D_{(k,j)} (v^{-i}) \]

In general, we have the following recursive definition.

**Definition 10.** For any solution \( \varphi \), we set
\[ \varphi_i (v) = 0 \quad \text{for all} \quad N \subseteq M \in \mathcal{N}, \, v \in \mathcal{V} (N), \, i \in M \setminus N. \quad (9) \]
Moreover, for all \( \alpha \in \mathbb{N}, \, N \subseteq M \in \mathcal{N}, \, v \in \mathcal{V} (N), \, i \in M, \) and \( i = (i_1, \ldots, i_\alpha) \in M^\alpha \), we define \( D_i \varphi (v) \) recursively by
\[
D_0 \varphi (v) := 0, \\
D_{(i)} \varphi (v) := \varphi_i (v), \\
D_i \varphi (v) := D_{(i_1, \ldots, i_\alpha - 1)} \varphi (v) - D_{(i_1, \ldots, i_\alpha - 1)} \varphi_i (v^{-i_\alpha}). \quad (10)
\]

The following theorem expresses a solution \( \varphi \) as the sum of higher-order contributions \( D_{(i,i)} \) of a fixed higher-order decomposer \( \varphi^{(\alpha)} \).

**Proposition 11.** For any decomposable solution \( \varphi \), we have
\[ \varphi_i (v) = \sum_{i \in N^\alpha} D_{(i,i)} \varphi^{(\alpha)} (v) \quad (11) \]
for all \( \alpha \in \mathbb{N}, \, N \in \mathcal{N}, \, v \in \mathcal{V} (N), \) and \( i \in N \).

To illustrate this result, let \( N = \{1, 2, 3\} \) and \( \alpha = 2 \). Then,
\[ \varphi_1 (v) = D_{(1,1,1)} \varphi^{(2)} (v) + D_{(1,2,1)} \varphi^{(2)} (v) + D_{(1,3,1)} \varphi^{(2)} (v) + D_{(2,1,1)} \varphi^{(2)} (v) + D_{(2,2,1)} \varphi^{(2)} (v) + D_{(2,3,1)} \varphi^{(2)} (v), \]

At first glance, this formula involves only third-order contributions. In view of Definition 10, however, some of the expressions actually are first-order contributions and second-order contributions. Indeed, we have
\[ \varphi_1 (v) = D_{(1)} \varphi^{(2)} (v) + 3 \cdot D_{(2,1)} \varphi^{(2)} (v) + 3 \cdot D_{(3,1)} \varphi^{(2)} (v) + D_{(2,3,1)} \varphi^{(2)} (v) + D_{(3,2,1)} \varphi^{(2)} (v). \]

In this formula, all (higher-order) contributions have a natural interpretation. In order to capture this observation more generally, we invoke the following definition.

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**Definition 12.** For all \( \alpha \in \mathbb{N} \) and \( N \in \mathcal{N} \) define the set of all sequences with members from \( N \) of length \( \alpha \) with pairwise different members by

\[
N^{[\alpha]} := \{ i \in N^\alpha \mid i_k \neq i_\ell \text{ for all } k, \ell \in \{1, \ldots, \alpha\} \},
\]

where \( N^\alpha \) denotes the set of all sequences with members from \( N \) of length \( \alpha \). Note that \( N^{[0]} = N^0 = \{()\} \) for all \( N \in \mathcal{N} \).

The following theorem expresses a solution \( \varphi \) as the sum of genuine (higher-order) contributions \( D_{(i,i)} \) of a fixed higher-order decomposer \( \varphi^{(\alpha)} \).

**Theorem 13.** Fix \( \alpha \in \mathbb{N} \). For any decomposable solution \( \varphi \), we have

\[
\varphi_i (v) = \sum_{t=0}^{\alpha} g(t, \alpha) \cdot \sum_{i \in (N \setminus \{i\})^{[t]}} D_{(i,i)} \varphi^{(\alpha)} (v)
\]

for \( N \in \mathcal{N}, v \in \mathcal{V} (N) \), and \( i \in N \), where \( g(t, \alpha) > 0 \) and

\[
g(t, \alpha) = \frac{1}{t !} \cdot \sum_{s=0}^{t} (-1)^{t-s} \cdot \binom{t}{s} \cdot (s + 1)^\alpha.
\]

Note that the summation index in (12) is in fact bounded above by \( \min \{ \alpha, n - 1 \} \). For \( t > n - 1 \), the right-most sum in (12) is empty because \( (N \setminus \{i\})^{[t]} = \emptyset \).

**6. Concluding remarks**

We introduce the decomposition of a solution by another solution (the decomposer) into a direct part and an indirect part. The indirect part indicates how much a player contributes to another player’s payoff according to the decomposer in a game. Taking this idea one step further leads to asking how much one player contributes to how much a second player contributes to a third player’s payoff in a game. This leads to the notion of higher-order contributions and higher-order decompositions of solutions. We characterize the solutions that can be decomposed and use this insight in order to provide a new foundation of the Shapley value. In particular, the Shapley value is the unique decomposable decomposer of the naïve solution, where the naïve solution assigns to any player her marginal contribution to all the other players.

**Appendix A. Proof of Theorem 3**

For all \( N \in \mathcal{N}, v \in \mathcal{V} (N) \), and \( i \in N \), we have

\[
\text{Sh}_i (v) + \sum_{\ell \in N \setminus \{i\}} \left( \text{Sh}_\ell (v) - \text{Sh}_\ell (v^{-i}) \right) = \sum_{\ell \in N} \text{Sh}_\ell (v) - \sum_{\ell \in N \setminus \{i\}} \text{Sh}_\ell (v^{-i})
\]

\[
= v (N) - v (N \setminus \{i\})
\]

\[
= N_{\text{Sh}} (v),
\]
i.e., the Shapley value is a decomposer of the naïve solution. Since the Shapley value admits a potential, Theorem 4 and Proposition 5 entail that the Shapley value is the first decomposer of the naïve solution.

Appendix B. Proof of Theorem 4

We show that decomposability and admittance of a potential are equivalent. The rest of the claims then drops from Calvo and Santos (1997, Proposition 3.4).

Let $P : \mathbb{V} \to \mathbb{R}$ be a potential for the solution $\varphi$. Let the solution $\psi^P$ be given by $\psi^P_i (v) = |N|^{-1} \cdot P (v)$ for all $N \in \mathcal{N}$, $v \in \mathbb{V} (N)$, and $i \in N$. By construction, we have

$$
\psi^P_i (v) + \sum_{\ell \in N \setminus \{i\}} \left( \psi^P_\ell (v) - \psi^P_\ell (v|N \setminus \{i\}) \right) = \sum_{\ell \in N} \psi^P_\ell (v) - \sum_{\ell \in N \setminus \{i\}} \psi^P_\ell (v|N \setminus \{i\}) \\
= P (v) - P (v^{-i}) \\
= \varphi_i (v)
$$

for all $N \in \mathcal{N}$, $v \in \mathbb{V} (N)$, and $i \in N$. That is, $\psi^P$ decomposes $\varphi$.

Let $\psi$ be a decomposer of the solution $\varphi$. Let the mapping $P^\psi : \mathbb{V} \to \mathbb{R}$ be given by $P^\psi (v) = \sum_{\ell \in N} \psi_\ell (v)$ for all $N \in \mathcal{N}$ and $v \in \mathbb{V} (N)$. By construction, we have

$$
P^\psi (v) - P^\psi (v|N \setminus \{i\}) = \sum_{\ell \in N \setminus \ell} \psi_\ell (v) - \sum_{\ell \in N \setminus \{i\}} \psi_\ell (v^{-i}) \\
= \psi_i (v) + \sum_{\ell \in N \setminus \{i\}} \left( \psi_\ell (v) - \psi_\ell (v^{-i}) \right) \\
= \varphi_i (v)
$$

for all $N \in \mathcal{N}$, $v \in \mathbb{V} (N)$, and $i \in N$. That is, $P^\psi$ is a potential for $\varphi$.

Appendix C. Proof of Proposition 5

Uniqueness: Let the solution $\varphi$ be decomposable and let $\psi$ and $\psi'$ be decomposable decomposers of $\varphi$. We show $\psi (v) = \psi' (v)$ for all $N \in \mathcal{N}$ and $v \in \mathbb{V} (N)$ by induction on $n$.

Induction basis: For $n = 1$, the claim is immediate from (4).

Induction hypothesis: Suppose $\psi (v) = \psi' (v)$ for all $N \in \mathcal{N}$ and $v \in \mathbb{V} (N)$ such that $n \leq t$.

Induction step: Let $N \in \mathcal{N}$ be such that $|N| = t + 1$. For $v \in \mathbb{V} (N)$ and $i \in N$, we have

$$
\psi_i (v) + \sum_{\ell \in N \setminus \{i\}} \left( \psi_\ell (v) - \psi_\ell (v^{-i}) \right) \overset{(4)}{=} \varphi_i (v) \overset{(4)}{=} \psi'_i (v) + \sum_{\ell \in N \setminus \{i\}} \left( \psi'_\ell (v) - \psi'_\ell (v^{-i}) \right).
$$

Hence, the induction hypothesis implies

$$
\sum_{\ell \in N} \psi_\ell (v) = \sum_{\ell \in N} \psi'_\ell (v). \quad \text{(C.1)}
$$
Since $\psi$ and $\psi'$ are decomposable and by Theorem 4, both $\psi$ and $\psi'$ satisfy the balanced contributions property. Hence, we have

$$\psi_j (v) - \psi_j (v^{-j}) = \psi_j (v) - \psi_j (v^{-i}) \quad \text{for all } j \in N \setminus \{i\}.$$  \hspace{1cm} (C.2)

Summing up (C.2) over all $j \in N \setminus \{i\}$ gives

$$(n - 1) \cdot \psi_i (v) - \sum_{j \in N \setminus \{i\}} \psi_j (v^{-j}) = \sum_{j \in N \setminus \{i\}} \left( \psi_j (v) - \psi_j (v^{-i}) \right)$$

and therefore

$$n \cdot \psi_i (v) = \sum_{\ell \in N} \psi_{\ell} (v) - \sum_{j \in N \setminus \{i\}} \psi_j (v^{-j}) + \sum_{j \in N \setminus \{i\}} \psi_j (v^{-j}).$$ \hspace{1cm} (C.3)

Analogously, we obtain

$$n \cdot \psi_i' (v) = \sum_{\ell \in N} \psi_{\ell}' (v) - \sum_{j \in N \setminus \{i\}} \psi_j' (v^{-j}) + \sum_{j \in N \setminus \{i\}} \psi_j' (v^{-j}).$$ \hspace{1cm} (C.4)

Finally, (C.1), (C.3), (C.4), and the induction hypothesis entail $\psi_i (v) = \psi_i' (v)$.

Existence: Let the solution $\varphi$ be decomposable and let the solution $\psi$ be given by (7). For all $N \in \mathcal{N}$, $v \in \mathbb{V} (N)$, and $i \in N$, we obtain

$$\psi_i (v) + \sum_{\ell \in N \setminus \{i\}} \left( \psi_{\ell} - \psi_{\ell} (v^{-i}) \right) = \sum_{\ell \in N} \psi_{\ell} (v) - \sum_{\ell \in N \setminus \{i\}} \psi_{\ell} (v^{-i})$$

$$(6),(7) \quad \sum_{T \in N : \ell \in T} \frac{\lambda_T (v^\ell)}{t}$$

$$(2) \quad \varphi_i (v),$$

where the last equation drops also invokes Theorem 4. Hence, $\psi$ is a decomposer of $\varphi$.

Finally, by Theorem 4, it suffices to show that $\psi$ admits a potential. In view (6) and (7), one easily checks that the mapping $P^\psi : \mathbb{V} \to \mathbb{R}$ given by

$$P^\psi (v) = \sum_{T \in N : T \neq \emptyset} \frac{\lambda_T (v^\ell)}{t^2}$$

for all $N \in \mathcal{N}$, and $v \in \mathbb{V} (N)$ is a potential for $\psi$.

**Appendix D. Proof of Proposition 11**

Let $\varphi$ be a decomposable solution. We proceed by induction on $\alpha \in \mathbb{N}$.

**Induction basis:** The claim is immediate for $\alpha = 0$.
For all $N \in \mathcal{N}$, $v \in \mathcal{V}(N)$, and $i \in N$, we have
\[
\sum_{j \in N} D_{(j,i)} \varphi^{(1)}(v) = \sum_{j \in N} [D_j \varphi^{(1)}(v) - D_j \varphi^{(1)}(v^{-i})] = \sum_{j \in N} \left[ \varphi_j^{(1)}(v) - \varphi_j^{(1)}(v^{-i}) \right] = \varphi_i^{(1)}(v) + \sum_{j \in N \setminus \{i\}} \left[ \varphi_j^{(1)}(v) - \varphi_j^{(1)}(v^{-i}) \right] = \varphi_i(v),
\]
where the last equation follows from the fact that $\varphi = \varphi^{(0)}$ is decomposed by $\varphi^{(1)}$.

**Induction hypothesis:** For all $\alpha \in \mathbb{N}$, $1 < \alpha \leq \bar{\alpha}$, $2 \leq \bar{\alpha}$, $N \in \mathcal{N}$, $v \in \mathcal{V}(N)$, and $i \in N$, we have
\[
\varphi_i(v) = \sum_{\ell \in N^\alpha} D_{(\ell,i)} \varphi^{(\alpha)}(v).
\]

**Induction step:** For all $N \in \mathcal{N}$, $v \in \mathcal{V}(N)$, and $i \in N$, we obtain
\[
\sum_{\ell \in N^{\bar{\alpha}+1}} D_{(\ell,i)} \varphi^{(\bar{\alpha}+1)}(v) = \sum_{\ell \in N^{\bar{\alpha}+1}} [D_{\bar{\alpha}+1} \varphi^{(\bar{\alpha}+1)}(v) - D_{\bar{\alpha}+1} \varphi^{(\bar{\alpha}+1)}(v^{-i})] = \sum_{j \in N} \varphi_j^{(1)}(v) - \sum_{j \in N \setminus \{i\}} \varphi_j^{(1)}(v^{-i}) = \varphi_i(v),
\]
where the second equation drops from the induction hypothesis, (9), and the fact that $\varphi^{(1)}$ is resolved by $\left((\varphi^{(\tau)})_{\tau \in \mathbb{N} \setminus \tau > 0}\right)$ and the third equation drops from the assumption that $\varphi^{(1)}$ decomposes $\varphi^{(0)} = \varphi$.

**Appendix E. Proof of Theorem 13**

We prove the claim by a number of lemmas.

**Lemma 14.** For any decomposable solution $\varphi$, we have
\[
D_{i} \varphi(v) = D_{\pi^i} \varphi(v)
\]
for all $\alpha \in \mathbb{N}$, $N \subseteq M \in \mathcal{N}$, $v \in \mathcal{V}(N)$, $i \in M^\alpha$, and all bijections $\pi : \{1, \ldots, \alpha\} \to \{1, \ldots, \alpha\}$, where $\pi^i \in M^\alpha$ is given by $(\pi^i)_\ell = i_{\pi(\ell)}$ for all $\ell \in \{1, \ldots, \alpha\}$.

**Proof.** Let $\varphi$ be a decomposable solution. We proceed by induction on $\alpha \in \mathbb{N}$.

**Induction basis:** The claim is immediate for $\alpha \leq 1$, and $i = (i, i), i \in M$. For $(i, j) \in M^{[2]}$, we obtain
\[
D_{(i,j)} \varphi(v) = \varphi_i(v) - \varphi_j(v^{-j}) = \varphi_j(v) - \varphi_j(v^{-i}) = D_{(j,i)} \varphi(v),
\]
for all $N \in \mathcal{N}$, $v \in \mathcal{V}(N)$, and $i \in N$. For $(i, j) \in M^{[2]}$, we obtain
\[
D_{(i,j)} \varphi(v) = \varphi_i(v) - \varphi_j(v^{-j}) = \varphi_j(v) - \varphi_j(v^{-i}) = D_{(j,i)} \varphi(v),
\]
for all $N \in \mathcal{N}$, $v \in \mathcal{V}(N)$, and $i \in N$. For $(i, j) \in M^{[2]}$, we obtain
\[
D_{(i,j)} \varphi(v) = \varphi_i(v) - \varphi_j(v^{-j}) = \varphi_j(v) - \varphi_j(v^{-i}) = D_{(j,i)} \varphi(v),
\]
where second equation follows from Theorem 4 and the assumption that \( \varphi \) is decomposable. This shows the claim for \( \alpha = 2 \).

**Induction hypothesis (IH):** Let the claim hold for all \( \alpha \in \mathbb{N}, \alpha \leq \bar{\alpha}, 2 \leq \bar{\alpha}, N \in \mathcal{N}, v \in \mathcal{V}(N), i \in M^\alpha, \) and all bijections \( \pi : \{1, \ldots, \alpha\} \to \{1, \ldots, \alpha\} \).

**Induction step:** For \( N \in \mathcal{N}, v \in \mathcal{V}(N), i \in N^{\bar{\alpha}}, i \in N, \) and all bijections \( \pi : \{1, \ldots, \bar{\alpha}\} \to \{1, \ldots, \bar{\alpha}\}, \) we obtain

\[
D_{(i,j)} \varphi (v) \overset{(10)}{=} D_i \varphi (v) - D_i \varphi (v^{-i}) \overset{IH}{=} D_\pi \varphi (v) - D_\pi \varphi (v^{-i}) \overset{(10)}{=} D_{(\pi(i),j)} \varphi (v).
\]

Further, for \( N \in \mathcal{N}, v \in \mathcal{V}(N), i \in M^{\bar{\alpha}-1}, \) and \( i, j \in M, \) equation (10) entails

\[
D_{(i,i,j)} \varphi (v) = D_{(i,i)} \varphi (v) - D_{(i,j)} \varphi (v^{-i})
\]

\[
= D_i \varphi (v) - D_i \varphi (v^{-i}) - (D_i \varphi (v^{-j}) - D_i \varphi (v^{-i-j}))
\]

\[
= D_{(i,j)} \varphi (v) - D_{(i,j)} \varphi (v^{-i})
\]

\[
= D_{(i,i,j)} \varphi (v).
\]

Hence, the claim holds for all bijections \( \pi : \{1, \ldots, \bar{\alpha} + 1\} \to \{1, \ldots, \bar{\alpha} + 1\} \).

For all \( \alpha \in \mathbb{N}, N \in \mathcal{N}, \) and \( i \in N^\alpha, \) we set \( \text{car}(i) := \{i_\ell | \ell \in \{1, \ldots, \alpha\}\} \). Hence, we have \( N^{\alpha} = \{i \in N^\alpha | |\text{car}(i)| = \alpha\}. \) Recall that \( N^{(0)} = N^0 = \{(0)\} \).

**Lemma 15.** For any decomposable solution \( \varphi, \) we have \( D_{(i,i)} \varphi (v) = D_i \varphi (v) \) for all \( \alpha \in \mathbb{N}, N \subseteq M \in \mathcal{N}, v \in \mathcal{V}(N), i \in M^\alpha, \) and \( i \in M \) such that \( i \in \text{car}(i) \).

**Proof.** Let \( \varphi \) be a decomposable solution and \( \alpha \in \mathbb{N}, N \subseteq M \in \mathcal{N}, v \in \mathcal{V}(N), i \in M^\alpha, \) and \( i \in N \) such that \( i \in \text{car}(i) \). By Lemma 14, we are allowed to assume that \( i_\alpha = i. \) By (10), we have

\[
D_{(i,i)} \varphi (v) = D_i \varphi (v) - D_i \varphi (v^{-i})
\]

\[
= D_i \varphi (v) - (D_{(i_1, \ldots, i_{\alpha-1})} \varphi (v^{-i}) - D_{(i_1, \ldots, i_{\alpha-1})} \varphi (v^{-i_{\alpha}}))
\]

\[
= D_i \varphi (v),
\]

which proves the claim. \( \square \)

The following lemma is immediate from Lemmas 14 and 15.

**Lemma 16.** If \( \varphi \) is decomposable, then \( D_i \varphi (v) = D_j \varphi (v) \) for all \( \alpha \in \mathbb{N}, N \subseteq M \in \mathcal{N}, v \in \mathcal{V}(N), \) and \( i, j \in M^\alpha \) such that \( \text{car}(i) = \text{car}(j) \).

For any decomposable solution \( \varphi \) and all \( N \in \mathcal{N}, v \in \mathcal{V}(N), S \in \mathcal{N}, \) we define higher-order differences \( D_S \varphi (v) \) by

\[
D_S \varphi (v) = D_i \varphi (v)
\]

with \( i \in M^\alpha, \alpha \in \mathbb{N} \) such that \( \text{car}(i) = S. \) In view of Lemma 16, \( D_S \varphi (v) \) is well-defined.
Lemma 17. For any decomposable solution \( \varphi \), we have

\[
\varphi_i(v) = \sum_{S \subseteq N \setminus \{i\}, \emptyset \leq s \leq \alpha} f(s, \alpha) \cdot D_{S \cup \{i\}} \varphi(\alpha)(v)
\]

for all \( \alpha \in \mathbb{N} \), \( N \in \mathcal{N} \), \( v \in \mathcal{V}(N) \), and \( i \in N \), where

\[
f(s, \alpha) = \sum_{t=0}^{s} \left( -1 \right)^{s-t} \binom{s}{t} \cdot (t + 1)^{\alpha}.
\]

(E.2)

Proof. The claim follows from Lemmas 11 and 16, and (E.1) as follows. For \( \alpha \in \mathbb{N} \), \( N \in \mathcal{N} \), \( S \subseteq N \), \( s \leq \alpha \), and \( i \in N \), set \( N_i^\alpha(S) = \{ i \in N^\alpha \mid \text{car}(i, i) = S \cup \{i\} \} \). The theorem then “claims” that \( |N_i^\alpha(S)| = f(s, \alpha) \). We prove the claim by induction on \( s \). Fix \( \alpha \).

Induction basis: For \( s = 0 \), i.e., \( S = \emptyset \), we have \( |N_i^\alpha(S)| = |\{(i, i, \ldots, i)\}| = 1 = f(0, \alpha) \).

Induction hypothesis (IH): Let the claim hold for all \( s \leq s < \alpha \).

Induction step: Fix \( S \subseteq N \setminus \{i\} \), \( |S| = s + 1 \). The number of \( i \in N^\alpha \) such that \( \text{car}(i, i) \subseteq S \cup \{i\} \) is \( (s + 1 + 1)^\alpha \). To obtain \( |N_i^\alpha(S)| \), we have to subtract the numbers \( |N_i^\alpha(T)| \), \( T \subsetneq S \). By the induction hypothesis, we have

\[
|N_i^\alpha(S)| \overset{IH}{=} (s + 1 + 1)^\alpha - \sum_{t=0}^{s} \binom{s + 1}{t} \cdot f(t, \alpha)
\]

(E.2)

\[
= (s + 1 + 1)^\alpha - \sum_{k=0}^{s} \binom{s + 1}{k} \cdot \sum_{t=0}^{k} \left( -1 \right)^{k-t} \binom{k}{t} \cdot (t + 1)^{\alpha}
\]

\[
= (s + 1 + 1)^\alpha - \sum_{t=0}^{s} (t + 1)^{\alpha} \sum_{k=t}^{s} \left( -1 \right)^{k-t} \binom{s + 1}{k} \cdot \binom{k}{t}
\]

\[
= (s + 1 + 1)^\alpha - \sum_{t=0}^{s} \binom{s + 1}{t} \cdot (t + 1)^{\alpha} \sum_{k=t}^{s} \left( -1 \right)^{k-t} \binom{s + 1 - t}{k-t}
\]

\[
= (s + 1 + 1)^\alpha - \sum_{t=0}^{s} \binom{s + 1}{t} \cdot (t + 1)^{\alpha} \sum_{k=0}^{s-t} \left( -1 \right)^{k} \binom{s + 1 - t}{k}
\]

\[
= (s + 1 + 1)^\alpha - \sum_{t=0}^{s} \binom{s + 1}{t} \cdot (t + 1)^{\alpha} \cdot (-1)^{s+1-t}
\]

\[
= (s + 1 + 1)^\alpha + \sum_{t=0}^{s} \binom{s + 1}{t} \cdot (t + 1)^{\alpha} \cdot (-1)^{s+1-t}
\]

\[
= \sum_{t=0}^{s+1} \left( -1 \right)^{s+1-t} \binom{s + 1}{t} \cdot (t + 1)^{\alpha},
\]

which concludes the proof.

The theorem now drops from Lemma 17 by the following fact. For all \( \alpha \in \mathbb{N} \), \( N \in \mathcal{N} \), \( i \in N \), and \( S \subseteq N \setminus \{i\} \) such that \( s \leq \alpha \), the set \( \{ i \in (N \setminus \{i\})^s \mid \text{car}(i) = S \} \) has a cardinality of \( s ! \). Finally, \( f(s, \alpha) = |N_i^\alpha(S)| > 0 \) implies \( g(s, \alpha) > 0 \).
References
