Weakly Balanced Contributions and the Weighted Shapley Values

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Abstract

We provide a concise characterization of the class of positively weighted Shapley values by three properties, two standard properties, efficiency and marginality, and a relaxation of the balanced contributions property called the weak balanced contributions property. Balanced contributions: the amount one player gains or loses when another player leaves the game equals the amount the latter player gains or loses when the former player leaves the game. Weakly balanced contributions: the direction (sign) of the change of one player’s payoff when another player leaves the game equals the direction (sign) of the change of the latter player’s payoff when the former player leaves the game. Since the (symmetric) Shapley value is characterized by efficiency and the balanced contributions property and also satisfies marginality, we pinpoint position of the Shapley value within the class of positively weighted Shapley values to obeying the balanced contributions property versus just obeying the weak balanced contributions property.

Keywords: TU game, weighted Shapley values, marginality, balanced contributions, weakly balanced contributions

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1. Introduction

The (symmetric) Shapley value (Shapley, 1953b) probably is the most eminent one-point solution concept for cooperative games with transferable utility (TU games or simply games). Besides its original axiomatic foundation by Shapley himself, alternative foundations of different types have been suggested later on. Important direct axiomatic characterizations are due to Myerson (1980) and Young (1985).

In order to account for asymmetries among players beyond the game itself, Shapley (1953a) already suggests weighted versions of his symmetric value, where these asymmetries are modelled by strictly positive weights for the players—the positively weighted Shapley

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There exist a number of axiomatic foundations for the whole class of positively weighted Shapley values (see, e.g., Kalai and Samet, 1987; Hart and Mas-Colell, 1989; Chun, 1991; Nowak and Radzik, 1995; Casajus, 2018, 2019).\(^1\)

Myerson’s (1980) characterization of the Shapley value involves just two properties, efficiency and the balanced contributions property. Efficiency: the worth of the grand coalition is distributed among the players. Balanced contributions: the amount one player gains or loses when another player leaves the game equals the amount the latter player gains or loses when the former player leaves the game. Since the positively weighted Shapley values satisfy efficiency, they fail the balanced contributions property with exception of the symmetric \(^3\) Shapley value. Instead, they satisfy the weak balanced contributions property: the direction (sign) of the change of one player’s payoff when another player leaves the game equals the direction (sign) of the change of the latter player’s payoff when the former player leaves the game (Casajus, 2017a).

Casajus (2017a, Lemma 1) already establishes some joint implications of the weak balanced contributions property and efficiency. In particular, he shows that these properties imply the dummy player property and the dummy player out property. Dummy player: all her marginal contributions to coalitions not containing her coincide with her singleton worth. Dummy player property: a dummy players’ payoff equals her singleton worth. Dummy player out property: removing a dummy player from a game doesn’t affect the remaining players’ payoffs. As our first result, we establish another joint implication of the weak balanced contributions property and efficiency (Proposition 1). In particular, they imply strong monotonicity: in any monotonic game, all payoffs are non-negative. In addition, a player’s payoff is zero if and only if she is a null player. Monotonic game: all marginal contributions are non-negative. Null player: a dummy player with zero singleton worth.

There exists a large class of solutions that satisfy efficiency and the weak balanced contributions property, among them not only the positively weighted Shapley values but a huge number of utterly implausible solutions (Casajus, 2017a). In particular, the latter solutions fail marginality due to Young (1985). Marginality: a player’s payoff only depends on her own productivity. As our main result, we show that the positively weighted Shapley values are exactly those solutions that satisfy efficiency, the weak balanced contributions property, and marginality (Theorem 3). That is, in presence of efficiency and marginality, the difference between the Shapley value and the positively weighted Shapley value boils down to obeying the balanced contributions property versus obeying just the weak balanced contributions property.

The remainder of this paper is organized as follows. In Section 2, we provide basic definitions and notation. In Section 3, we discuss the weak balanced contributions property. In Section 4, we provide our characterization of the class of positively weighted Shapley values.

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\(^1\)Later on, Kalai and Samet (1987) extend the class of positively weighted Shapley values by considering weight systems that allow for zero weights of the players.

\(^2\)Characterizations for individual positively weighted Shapley values have been suggested by Hart and Mas-Colell (1989) or Calvo and Santos (2000), for example.

\(^3\)From now on, we refrain from mentioning this whenever it applies.
values. In Section 5, we conclude our paper with a comparison of our characterization with the recent characterization suggested by Casajus (2018). An appendix contains the proof of our main result.

2. Basic definitions and notation

Let $\mathcal{U}$ be a countably infinite set, the universe of players, and let $\mathcal{N}$ denote the set of all finite subsets of $\mathcal{U}$. A (finite TU) game for the player set $N \in \mathcal{N}$ is given by a coalition function $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$, where $2^N$ denotes the power set of $N$. Subsets of $N$ are called coalitions; $v(S)$ is called the worth of coalition $S$. The set of all games for $N$ is denoted by $\mathcal{V}(N)$; the set of all subgames for $N$ is denoted by $\mathcal{V}_\subseteq(N) := \bigcup_{T \subseteq N} \mathcal{V}(T)$; the set of all games is denoted by $\mathcal{V} := \bigcup_{N \in \mathcal{N}} \mathcal{V}(N)$.

For $N \in \mathcal{N}$, $T \subseteq N$, and $v \in \mathcal{V}(N)$, the coalition function $v|_T \in \mathcal{V}(T)$ is given by $v|_T(S) = v(S)$ for all $S \subseteq T$. The game $0^N \in \mathcal{V}(N)$ given by $0^N(S) = 0$ for all $S \subseteq N$ is called the null game. For $N \in \mathcal{N}$, $v,w \in \mathcal{V}(N)$, and $\alpha \in \mathbb{R}$, the coalition functions $v+w \in \mathcal{V}(N)$ and $\alpha \cdot v \in \mathcal{V}(N)$ are given by $v(S) = v(S) + w(S)$ and $(\alpha \cdot v)(S) = \alpha \cdot v(S)$ for all $S \subseteq N$. For $T \subseteq N$, $T \neq \emptyset$, the game $u^T_T \in \mathcal{V}$ given by $u^T_T(S) = 1$ if $T \subseteq S$ and $u^T_T(S) = 0$ otherwise is called a unanimity game. Any $v \in \mathcal{V}(N)$, $N \in \mathcal{N}$ can be uniquely represented by unanimity games. In particular, we have

$$v = \sum_{T \subseteq N : T \neq \emptyset} \lambda_T(v) \cdot u^N_T,$$

where the coefficients $\lambda_T(v)$ are known as the Harsanyi dividends (Harsanyi, 1959) and can be determined recursively by

$$\lambda_T(v) := v(T) - \sum_{S \subseteq T : S \neq \emptyset} \lambda_S(v).$$

A game $v \in \mathcal{V}(N)$, $N \in \mathcal{N}$ is called monotonic, if $v(S) \leq v(T)$ for all $S,T \subseteq N$ such that $S \subseteq T$. Player $i \in N$ is called a dummy player in $v \in \mathcal{V}(N)$ if $v(S \cup \{i\}) - v(S) = v(\{i\})$ for all $S \subseteq N \setminus \{i\}$; if in addition $v(\{i\}) = 0$, player $i$ is called a null player; players $i,j \in N$ are called symmetric in $v \in \mathcal{V}(N)$ if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i,j\}$.

A solution/value for a subset $V$ of $\mathcal{V}$ is an operator that assigns to any $N \in \mathcal{N}$, $v \in \mathcal{V}(N) \cap V$, and $i \in N$ a payoff $\varphi_i(v)$. The (symmetric) Shapley value (Shapley, 1953b) for $V \subseteq \mathcal{V}$, $\text{Sh}$, is given by

$$\text{Sh}_i(v) := \sum_{T \subseteq N : i \in T} |T|^{-1} \cdot \lambda_T(v) \quad \text{for all } N \in \mathcal{N}, v \in \mathcal{V}(N) \cap V, \text{ and } i \in N. \quad (3)$$

Let $\mathbb{R}_{++}^\mathcal{U} := \{f : \mathcal{U} \to \mathbb{R}_{++}\}$ and $w_i := w(i)$ for all $w \in \mathbb{R}_{++}^\mathcal{U}$ and $i \in \mathcal{U}$. For $w \in \mathbb{R}_{++}^\mathcal{U}$, the $w$-weighted Shapley value for $V \subseteq \mathcal{V}$, $\text{Sh}_w^w$, is given by

$$\text{Sh}_i^w(v) := \sum_{T \subseteq N : i \in T} \frac{w_i}{\sum_{\ell \in T} w_\ell} \cdot \lambda_T(v) \quad \text{for all } N \in \mathcal{N}, v \in \mathcal{V}(N) \cap V, \text{ and } i \in N. \quad (4)$$
The solutions $Sh^w$, $w \in \mathbb{R}_{++}^N$, are called the **positively weighted Shapley values** (Shapley, 1953a).

In the following, we make use of the following (standard) properties of solutions. The restriction of a property to a subset $V$ of $V$ is indicated by the suffix “$|_V$”; instead of “$|_V(N)$”, $N \in N$, we write “$|_N$”.

**Efficiency, E.** For all $N \in N$ and $v \in V(N)$, we have $\sum_{\ell \in N} \varphi_\ell (v) = v(N)$.

**Additivity, A.** For all $N \in N$ and $v, w \in V(N)$, we have $\varphi_\ell (v + w) = \varphi_\ell (v) + \varphi_\ell (w)$ for all $\ell \in N$.

**Symmetry, S.** For all $N \in N$, $v \in V(N)$, and $i, j \in N$ such that $i$ and $j$ are symmetric in $v$, we have $\varphi_i (v) = \varphi_j (v)$.

**Null player, N.** For all $N \in N$, $v \in V(N)$, and $i \in N$ such that $i$ is a null player in $v$, we have $\varphi_i (v) = 0$.

**Null player out, NPO.** For all $N \in N$, $v \in V(N)$, and $i \in N$ such that $i$ is a null player in $v$, we have $\varphi_i (v) = \varphi_{\ell |_{N \setminus \{i\}}} (v)$ for all $\ell \in N \setminus \{i\}$.

**Dummy player, D.** For all $N \in N$, $v \in V(N)$, and $i \in N$ such that $i$ is a dummy player in $v$, we have $\varphi_i (v) = v(\{i\})$.

**Dummy player out, DPO** (Tijs and Driessen, 1986). For all $N \in N$, $v \in V(N)$, and $i \in N$ such that $i$ is a dummy player in $v$, we have $\varphi_\ell (v) = \varphi_\ell (v|_{N \setminus \{i\}})$ for all $\ell \in N \setminus \{i\}$.

**Positivity, P** (Kalai and Samet, 1987). For all $N \in N$, $v \in V(N)$, and $i \in N$ such that $v$ is monotonic, we have $\varphi_i (v) \geq 0$.

**Strict positivity, P**' (Kalai and Samet, 1987). For all $N \in N$, $v \in V(N)$, and $i \in N$ such that $v$ is monotonic and such that there are no null players in $v$, we have $\varphi_i (v) > 0$ for all $i \in N$.

**Marginality, M** (Young, 1985). For all $N \in N$, $v, w \in V(N)$, and $i \in N$ such that $v(S \cup \{i\}) - v(S) = w(S \cup \{i\}) - w(S)$ for all $S \subseteq N \setminus \{i\}$, we have $\varphi_i (v) = \varphi_i (w)$.

**Strong monotonicity, Mo** (Young, 1985). For all $N \in N$, $v, w \in V(N)$, and $i \in N$ such that $v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S)$ for all $S \subseteq N \setminus \{i\}$, we have $\varphi_i (v) \geq \varphi_i (w)$.

### 3. Weakly balanced contributions

Myerson (1980) provides an important characterization of the Shapley value for games with a fixed player set and all their subgames by two properties: efficiency and the balanced contributions property.4

**Balanced contributions, BC.** For all $N \in N$, $v \in V(N)$, $i \in N$, and $j \in N \setminus \{i\}$, we have

$$\varphi_i (v) - \varphi_i (v|_{N \setminus \{i\}}) = \varphi_j (v) - \varphi_j (v|_{N \setminus \{i\}})$$

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4The balanced contributions property is equivalent to a number properties: path independence (Hart and Mas-Colell, 1989), consistency with the Shapley value (Calvo and Santos, 1997), admittance of a potential (Calvo and Santos, 1997; Ortmann, 1998), and decomposability and resolvability (Casajus and Huetttner, 2018).
This property requires player $j$’s contribution to the payoff of player $i$, $\varphi_i(v) - \varphi_i(v|N\setminus\{j\})$, to equal player $i$’s contribution to the payoff of player $j$, $\varphi_j(v) - \varphi_j(v|N\setminus\{i\})$. In this sense, the players’ mutual contributions are balanced (quantitatively).

From this characterization it is clear that the positively weighted Shapley values fail the balanced contributions property. Instead, they satisfy a relaxation of the balanced contributions property—the weak balanced contributions property (Casajus, 2017a).\footnote{Yokote and Kongo (2017) and Kongo (2018) suggest alternative relaxations of the balanced contributions property.} Recall the sign function $\text{sign}: \mathbb{R} \to \{-1, 0, 1\}$ given by $\text{sign}(\xi) = 1$ for $\xi > 0$, $\text{sign}(0) = 0$, and $\text{sign}(\xi) = -1$ for $\xi < 0$.

**Weak balanced contributions, BC$^-$.** For all $N \in \mathcal{N}$, $v \in \mathcal{V}(N)$, $i \in N$, and $j \in N\setminus\{i\}$, we have

$$\text{sign} \left( \varphi_i(v) - \varphi_i(v|N\setminus\{j\}) \right) = \text{sign} \left( \varphi_j(v) - \varphi_j(v|N\setminus\{i\}) \right).$$

This property requires player $j$’s contribution to the payoff of player $i$, $\varphi_i(v) - \varphi_i(v|N\setminus\{j\})$, to be greater (less) than zero if and only if player $i$’s contribution to the payoff of player $j$, $\varphi_j(v) - \varphi_j(v|N\setminus\{i\})$, is greater (less) than zero. In this sense, the players’ mutual contributions are weakly balanced or balanced qualitatively.

Casajus (2017a, Lemma 1) shows that the weak balanced contributions property together with efficiency implies the dummy player property and the dummy player out property. We now show that the former two properties jointly also entail a strong positivity property. Note that strong positivity is stronger than both positivity and strict positivity.

**Strong positivity, P$^+$.** For all $N \in \mathcal{N}$, $v \in \mathcal{V}(N)$, and $i \in N$ such that $v$ is monotonic, we have (i) $\varphi_i(v) \geq 0$ and (ii) $\varphi_i(v) = 0$ if and only if $i$ is a null player in $v$.

**Proposition 1.** If a solution satisfies efficiency (E) and the weak balanced contributions property (BC$^-$), then it satisfies strong positivity (P$^+$).

It is immediate that efficiency implies strong positivity for one-player games. So, the weak balanced contributions property lifts the latter to games with arbitrary player sets.

**Proof.** Let the solution $\varphi$ be as in the proposition. By Casajus (2017a, Lemma 1), this solution satisfies $\mathbf{N}$. We show the claim by induction on $|N|$.

**Induction basis:** The claim is immediate for all $N \in \mathcal{N}$ and $v \in \mathcal{V}(N)$ such that $|N| \leq 1$.

**Induction hypothesis (IH):** Suppose the claim holds for all $N \in \mathcal{N}$ and $v \in \mathcal{V}(N)$ such that $|N| \leq t$.

**Induction step:** Let $N \in \mathcal{N}$ and $v \in \mathcal{V}(N)$ be such that $|N| = t + 1$ and that $v$ is monotonic. Suppose (\*) $\varphi_i(v) < 0$ for some $i \in N$. This implies

$$\varphi_i(v) \overset{\text{IH}}{<} \varphi_i(v|N\setminus\{j\}) \quad \text{for all } j \in N\setminus\{i\} \quad (5)$$

and therefore

$$\varphi_j(v) \overset{\text{BC}^-}{<} \varphi_j(v|N\setminus\{i\}) \quad \text{for all } j \in N\setminus\{i\}. \quad (6)$$
Summing up (6) over \( j \in N \setminus \{i\} \) gives

\[
v(N) \overset{E}{=} \sum_{j \in N} \varphi_j(v) < \sum_{j \in N \setminus \{i\}} \varphi_j(v) \overset{(*)}{=} v(N \setminus \{i\}),
\]

contradicting \( v \) being monotonic. Hence, we have \( \varphi_i(v) \geq 0 \) for all \( j \in N \).

Assume now (**) \( \varphi_i(v) = 0 \). Then, (***) the strict inequalities in (5) and (6) become weak ones. Analogously to (7), we obtain

\[
v(N) \overset{E}{=} \sum_{j \in N} \varphi_j(v) \overset{(**)}{=} \sum_{j \in N \setminus \{i\}} \varphi_j(v) \overset{E}{=} v(N \setminus \{i\}). \tag{8}
\]

Since \( v \) is monotonic, the inequality in (8) becomes an equality. Hence, we have

\[
v(N) = v(N \setminus \{i\}) \tag{9}
\]

and

\[
\varphi_j(v) \overset{(**)}{=} \varphi_j(v|_{N \setminus \{i\}}) \quad \text{for all} \quad j \in N \setminus \{i\}. \tag{10}
\]

By BC\(^{-}\), we thus have

\[
\varphi_i(v) = \varphi_i(v|_{N \setminus \{i\}}) \quad \text{for all} \quad j \in N \setminus \{i\}. \tag{11}
\]

Moreover, we have

\[
\varphi_i(v) \overset{E}{=} v(N) - \sum_{j \in N \setminus \{i\}} \varphi_j(v) \overset{E}{=} 0. \tag{12}
\]

By (11), (12), and the induction hypothesis, player \( i \) is a null player in \( v|_{N \setminus \{j\}} \) for all \( j \in N \setminus \{i\} \). In view of (9), player \( i \) is a null player in \( v \), which concludes the proof. \( \square \)

### 4. The positively weighted Shapley values

In this section, cum grano salis, we pinpoint the position of the Shapley value within the class of positively weighted Shapley values to obeying the balanced contributions property versus just obeying the weak balanced contributions property. It is clear that Myerson’s (1980) characterization of the Shapley value also works for solutions for \( V \).

**Theorem 2 (Myerson, 1980).** The Shapley value for \( V \) is the unique solution that satisfies efficiency (E) and the balanced contributions property (BC).

As our main result, we obtain the following characterization of the class of positively weighted Shapley values. The proof of the theorem is referred to Appendix A.

**Theorem 3.** A solution \( \varphi \) for \( V \) satisfies efficiency (E), marginality (M), and the weak balanced contributions property (BC\(^{-}\)) if and only if \( \varphi = Sh^w \) for some \( w \in \mathbb{R}_{++} \).
Remark 4. Kalai and Samet (1987, Theorem 1) show that the weighted Shapley values are probabilistic, which implies that they also satisfy strong monotonicity. Hence, in Theorem 3, marginality can be replaced with strong monotonicity.

Remark 5. As Casajus (2018) and other than Kalai and Samet (1987), Hart and Mas-Colell (1989), Chun (1991), Nowak and Radzik (1995), and (Casajus, 2019), our characterization assumes a countably infinite universe of players as in Shapley’s original paper (Shapley, 1953b). Whereas this assumption is crucial for Casajus (2018), careful inspection of the proof of Theorem 3 reveals that our characterization also works for solutions for \( V_{\subseteq} (N) \), \( N \in \mathcal{N} \) if \( |N| > 3 \).

Remark 6. This characterization is non-redundant. The nil solution, \( Nil \), given by \( Nil_i (v) = 0 \) for all \( N \in \mathcal{N} \), \( v \in V (N) \), and \( i \in N \) satisfies all axioms but efficiency. Let \( \Omega := \{ f : \mathbb{R} \times \mathcal{U} \rightarrow \mathbb{R}_+ \} \) and let the solutions \( \varphi^\omega \), \( \omega \in \Omega \) be given by

\[
\varphi^\omega_i (v) := \sum_{T \subseteq N; i \in T} \sum_{\ell \in T} \omega (v (\{ \ell \}), i) \cdot \lambda_T (v) \quad \text{for all } N \in \mathcal{N}, v \in V (N), \text{ and } i \in N.
\]

These solutions satisfy efficiency and the weak balanced contributions property in general and marginality if and only if \( \omega \in \Omega \) is insensitive to its first argument, i.e., \( \omega (\alpha, i) = \omega (\beta, i) \) for all \( \alpha, \beta \in \mathbb{R} \) and \( i \in \mathcal{U} \) (Casajus, 2017b). Let \( \rho : \mathcal{U} \rightarrow \mathbb{N} \) be injective and the solution \( \varphi^\rho \) be given by

\[
\varphi^\rho_i (v) = v (\{ \ell \in N \mid \rho (\ell) \leq \rho (i) \}) - v (\{ \ell \in N \mid \rho (\ell) < \rho (i) \})
\]

for all \( N \in \mathcal{N}, v \in V (N), \text{ and } i \in N \). This solution satisfies all axioms but the weak balanced contributions property.

Remark 7. A solution \( \varphi^\omega \), \( \omega \in \Omega \) satisfies efficiency, marginality, and the weak balanced contributions property if and only if \( \varphi^\omega = Sh^w \) for some \( w \in \mathbb{R}_+^u \). Since it is neither obvious nor clear at all that the solutions \( \varphi^\omega \), \( \omega \in \Omega \) are all solutions that satisfy efficiency and the weak balanced contributions property, Casajus (2017b) does not already provide a proof of Theorem 3.

5. Discussion

In this section, we compare our new characterization of the class of positively weighted Shapley values with the equally concise characterization due to Casajus (2018). For a discussion of the characterizations by Kalai and Samet (1987), Hart and Mas-Colell (1989), Chun (1991), and Nowak and Radzik (1995), we refer the readers to Casajus (2018, Section 5).

The crucial axiom in Casajus’ (2018) characterization is the following one. For a discussion of this property, we refer the reader to the original paper.

Superweak differential marginality, \( \text{DM}^- \). For all \( N \in \mathcal{N}, i, j \in N, \text{ and } v, w \in V (N) \) such that

\[
v (S \cup \{ i \}) - v (S) = w (S \cup \{ i \}) - w (S)
\]
and
\[ v(S \cup \{j\}) - v(S) = w(S \cup \{j\}) - w(S) \]
for all \( S \subseteq N \setminus \{i, j\} \), we have
\[ \text{sign}(\varphi_i(v) - \varphi_i(w)) = \text{sign}(\varphi_j(v) - \varphi_j(w)). \]
Together with efficiency and the null player out property, this property characterizes the class of positively weighted Shapley values (Casajus, 2018, Theorem 6).

Since both collections of axioms (i) efficiency, marginality, and the weak balanced contributions property and (ii) efficiency, the null player out property, and superweak differential marginality characterize the class of positively weighted Shapley values, it is clear that either of the two sets implies the other one. On the one hand, there are short and direct proofs that the null player property, the null player out property, and superweak differential marginality jointly imply both marginality and the weak balanced contributions property (Casajus, 2018, Propositions 2 and 3). Since the null player property is immediate from efficiency and the null player out property, one can directly and rather easily infer collection (i) of axioms from collection (ii).

On the other hand, there also is a short and direct proof that efficiency and the weak balanced contributions property jointly imply the null player out property (Casajus, 2017b, Lemma 1), but there seems to be no easy way to infer superweak differential marginality from collection (i) of axioms. Thus, in some informal sense, collection (i) of axioms is weaker than collection (ii) of axioms. This is also reflected by the fact that the proof of Theorem 3 is much more involved than the proof of Casajus (2018, Theorem 6).

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Appendix A. Proof of Theorem 3

It is well-known that the solutions \( \text{Sh}^w \), \( w \in \mathbb{R}^{1+} \) satisfy \( \text{E} \). Kalai and Samet (1987, Theorem 1) and Casajus (2017a) show that they satisfy \( \text{M} \) and \( \text{BC}^- \), respectively. Let the solution \( \varphi \) for \( V \) satisfy \( \text{E} \), \( \text{M} \), and \( \text{BC}^- \). Casajus (2017a, Lemma 1) shows that \( \text{E} \) and \( \text{BC}^- \) imply \( \text{N} \) and \( \text{NPO} \). We show that \( \varphi = \text{Sh}^w \) for some \( w \in \mathbb{R}^{1+} \) using a number of claims.

For \( N \in \mathcal{N} \) and \( v \in \mathbb{V}(N) \), two players \( i, j \in N \), \( i \neq j \) are called mutually dependent in \( v \), if \( v(S \cup \{i\}) = v(S) = v(S \cup \{j\}) \) for all \( S \subseteq N \setminus \{i, j\} \), i.e., if they are only jointly productive (Nowak and Radzik, 1995). One can easily check that players \( i \) and \( j \) are mutually dependent in \( v \) if and only if \( \lambda_T(v) = 0 \) for all \( T \subseteq N \) such that \( |T \cap \{i, j\}| = 1 \).

Claim 1, \textbf{C01}: For all \( N \in \mathcal{N} \), \( v \in \mathbb{V}(N) \), and \( i, j \in N \), \( i \neq j \) such that \( i \) and \( j \) are mutually dependent in \( v \), we have \( \text{sign}(\varphi_i(v)) = \text{sign}(\varphi_j(v)) \).\footnote{Casajus (2019) calls this property \textit{weak sign symmetry}. Together with efficiency, additivity, and the null player property, it characterizes the class of positively weighted Shapley values.
If $i$ and $j$ are mutually dependent in $v$, then (*) $i$ and $j$ are null players in $v_{|N\setminus\{j\}}$ and $v_{|N\setminus\{i\}}$, respectively. We obtain

$$\text{sign} \left( \varphi_i (v) \right) \stackrel{(\ast) N}{=} \text{sign} \left( \varphi_i (v) - \varphi_i (v_{|N\setminus\{j\}}) \right)$$

$$\equiv \text{sign} \left( \varphi_j (v) - \varphi_j (v_{|N\setminus\{i\}}) \right) \stackrel{(\ast) N}{=} \text{sign} \left( \varphi_j (v) \right).$$

**Claim 2, C02:** For all $N \in \mathcal{N}$, $v \in \mathcal{V}(N)$, and $P \subseteq N$ such that any two players in $P$ are mutually dependent in $v$,\footnote{Kalai and Samet (1987) call such a coalition a coalition of partners.} we have

$$\varphi_i \left( v - \left( \sum_{\ell \in P} \varphi_\ell (v) \right) \cdot u_P^N \right) = 0 \quad \text{for all} \quad i \in P.$$

Set $w = \sum_{\ell \in P} \varphi_\ell (v) \cdot u_P^N$. By M, we have $\varphi_\ell (v) = \varphi_\ell (v - w)$ for all $\ell \in N \setminus P$. By E, we obtain $\sum_{\ell \in P} \varphi_\ell (v - w) = 0$. Since any two players in $P$ are mutually dependent in $v - w$, the claim drops from C01.

For all $N \in \mathcal{N}$ and $v \in \mathcal{V}(N)$, set

$$T(v) := \{ T \subseteq 2^N \setminus \{ \emptyset \} | \lambda_T (v) \neq 0 \} \quad \text{(A.1)}$$

and

$$C(v) := \bigcup_{T \in T(v)} T. \quad \text{(A.2)}$$

**Claim 3, C03:** For all $N \in \mathcal{N}$ and $v, w, z \in \mathcal{V}(N)$, we have

$$\sum_{\ell \in C(\pi) \setminus C(z)} \varphi_\ell (v + w + z)$$

$$= \sum_{\ell \in C(\pi) \cap C(z)} \varphi_\ell (v + w) + \sum_{\ell \in C(\pi) \cap C(z)} \varphi_\ell (v + z) - \sum_{\ell \in C(\pi) \setminus C(z)} \varphi_\ell (v).$$

We obtain

$$\sum_{\ell \in C(\pi) \setminus C(z)} \varphi_\ell (v + w + z)$$

$$\equiv (v + w + z)(N)$$

$$= \sum_{\ell \in C(\pi) \setminus C(z)} \varphi_\ell (v + w) - \sum_{\ell \in C(\pi) \setminus C(z)} \varphi_\ell (v + z) - \sum_{\ell \in N \setminus (C(\pi) \cup C(z))} \varphi_\ell (v)$$

$$= -v(N) + v(N) + w(N) - \sum_{\ell \in C(\pi) \setminus C(z)} \varphi_\ell (v + w)$$

$$+ v(N) + z(N) - \sum_{\ell \in C(\pi) \setminus C(z)} \varphi_\ell (v + z) - \sum_{\ell \in N \setminus (C(\pi) \cup C(z))} \varphi_\ell (v).$$
\[ E = -v(N) + \sum_{\ell \in C(w) \cap C(z)} \varphi_{\ell}(v + w) + \sum_{\ell \in N \setminus C(w)} \varphi_{\ell}(v + z) + \sum_{\ell \in N \setminus (C(w) \cup C(z))} \varphi_{\ell}(v) \]

\[ = -v(N) + \sum_{\ell \in C(w) \cap C(z)} \varphi_{\ell}(v + w) + \sum_{\ell \in N \setminus C(w)} \varphi_{\ell}(v) + \sum_{\ell \in C(w) \cap C(z)} \varphi_{\ell}(v + z) - \sum_{\ell \in N \setminus (C(w) \cup C(z))} \varphi_{\ell}(v) \]

which proves the claim.

**Claim 4, C04:** For all \( N \in \mathcal{N}, v, w \in \mathbb{V}(N), \) and \( i, j \in N \) such that \( i \in C(w), j \notin C(w), \) and \( i \) and \( j \) are mutually dependent in \( v, \) we have

\[ \varphi_i(v + w) = \varphi_i(v) + \varphi_i(w). \]

Set

\[ \lambda_{ij}^* := \varphi_i(v) + \varphi_j(v). \] (A.3)

We have

\[ \varphi_i(v) = \varphi_i(v + w - \lambda_{ij}^* \cdot u_{i,j}^N) \]

\[ \overset{\text{C03}}{=} \varphi_i(v + w) + \varphi_i(v - \lambda_{ij}^* \cdot u_{i,j}^N) - \varphi_i(v) \]

\[ \overset{(A.3), \text{C02}}{=} \varphi_i(v + w) - \varphi_i(v), \]

where the first equation follows from

\[ \varphi_j(v + w - \lambda_{ij}^* \cdot u_{i,j}^N) \overset{\text{M}}{=} \varphi_j(v - \lambda_{ij}^* \cdot u_{i,j}^N) \]

\[ \overset{(A.3), \text{C02}}{=} 0 \]

\[ \overset{\text{N}}{=} \varphi_j(0^{N \setminus \{i\}}) \]

\[ \overset{\text{M}}{=} \varphi_j((-v + w - \lambda_{ij}^* \cdot u_{i,j}^N) \mid_{N \setminus \{i\}}), \]

which implies

\[ \varphi_i(v + w - \lambda_{ij}^* \cdot u_{i,j}^N) \overset{\text{BC}}{=} \varphi_i((-v + w - \lambda_{ij}^* \cdot u_{i,j}^N) \mid_{N \setminus \{j\}}) \]

\[ \overset{\text{M}}{=} \varphi_i(w \mid_{N \setminus \{j\}}) \]

\[ \overset{\text{NPO}}{=} \varphi_i(w). \]
Claim 5, C05: For all $N \in \mathcal{N}$, $n > 2$ and $i \in N$, the mapping $r_i^N : \mathbb{R} \to \mathbb{R}$ given by

$$r_i^N (\rho) = \varphi_i (\rho \cdot u_N^i) \quad \text{for all } \rho \in \mathbb{R} \quad \text{(A.4)}$$

is monotonically increasing.

Let $\rho, \rho^+ \in \mathbb{R}$, $\rho^+ > \rho$. Suppose (*) $r_i^N (\rho^+) < r_i^N (\rho)$. By (A.4) and E, there is some $j \in N \setminus \{i\}$ such that (**) $\varphi_j (\rho^+ \cdot u_N^j) > \varphi_j (\rho \cdot u_N^j)$. Set

$$w = -(\varphi_i (\rho \cdot u_N^i) + \varphi_j (\rho \cdot u_N^j)) \cdot u_N^{i,j} \quad \text{(A.5)}$$

We obtain

$$\varphi_j (\rho^+ \cdot u_N^j + w) \stackrel{C04}{=} \varphi_j (\rho^+ \cdot u_N^j) + \varphi_j (w) \stackrel{(**)}{>} \varphi_j (\rho \cdot u_N^j) + \varphi_i (w) \stackrel{C04}{=} \varphi_j (\rho \cdot u_N^j + w) \stackrel{(A.5), C02}{=} 0.$$ 

Since $i$ and $j$ are mutually dependent in $w$, we have (***) $0 < \varphi_i (\rho^+ \cdot u_N^i + w)$. We obtain

$$\varphi_i (\rho^+ \cdot u_N^i) + \varphi_i (w) \stackrel{C04}{=} \varphi_i (\rho^+ \cdot u_N^i + w) \stackrel{(***)}{>} 0 \stackrel{(A.5), C02}{=} \varphi_i (\rho \cdot u_N^i + w) \stackrel{C04}{=} \varphi_i (\rho \cdot u_N^i) + \varphi_i (w),$$

i.e.,

$$r_i^N (\rho^+) \stackrel{(A.4)}{=} \varphi_i (\rho^+ \cdot u_N^i) > \varphi_i (\rho \cdot u_N^i) \stackrel{(A.4)}{=} r_i^N (\rho),$$

which contradicts (*).

Claim 6, C06: For all $N \in \mathcal{N}$, $n > 2$ and $i \in N$, the mapping $r_i^N : \mathbb{R} \to \mathbb{R}$ given by (A.4) is continuous.

Suppose $r_i^N$ is discontinuous at $\rho \in \mathbb{R}$. Then, there exists a sequence $(\rho_t)_{t \in \mathbb{N}}$, $\rho_k \in \mathbb{R}$ such that (*) $\lim_{t \to \infty} \rho_t = \rho$ and $\lim_{t \to \infty} r_i^N (\rho_t) \neq r_i^N (\rho)$. Suppose $F^+ := \lim_{t \to \infty} r_i^N (\rho_k) > r_i^N (\rho)$. There exist some $t_1 \in \mathbb{N}$ such that

$$\varphi_i (\rho_t \cdot u_N^i) \stackrel{(A.4)}{=} r_i^N (\rho_t) \geq r_i^N (\rho) + \varepsilon \stackrel{(A.4)}{=} \varphi_i (\rho \cdot u_N^i) + \varepsilon \quad \text{for all } t \geq t_1, \quad \text{(A.6)}$$

where $\varepsilon := \frac{1}{2} \cdot (F^+ - r_i^N (\rho)) > 0$. By C05, we have (***) $\rho \leq \rho_t$ for all $t \geq t_1$. By (*), there exists some $t_2 \in \mathbb{N}$, $t_2 \geq t_1$ such that (***) $\rho_t \geq \rho + \frac{\varepsilon}{2}$ for all $t \geq t_2$. For $t^* \in \mathbb{N}$, $t^* \geq t_2$, we
have
\[
\begin{align*}
\rho + \varepsilon & \geq \frac{\rho}{2} + \rho_i^* \\
& \equiv \sum_{\ell \in N} \varphi_\ell \left( \rho_i^* \cdot u_i^N \right) \\
& \geq \varphi_i \left( \rho \cdot u_i^N \right) + \varepsilon + \sum_{\ell \in N \setminus \{i\}} \varphi_\ell \left( \rho_i^* \cdot u_i^N \right) \\
& \equiv \rho - \sum_{\ell \in N \setminus \{i\}} \varphi_\ell \left( \rho \cdot u_i^N \right) + \varepsilon + \sum_{\ell \in N \setminus \{i\}} \varphi_\ell \left( \rho_i^* \cdot u_i^N \right)
\end{align*}
\]
and therefore
\[
\sum_{\ell \in N \setminus \{i\}} \varphi_\ell \left( \rho \cdot u_i^N \right) \geq \frac{\varepsilon}{2} + \sum_{\ell \in N \setminus \{i\}} \varphi_\ell \left( \rho_i^* \cdot u_i^N \right).
\]
This entails that there exists some \( \ell \in N \setminus \{i\} \) such that \( \varphi_\ell \left( \rho \cdot u_i^N \right) > \varphi_\ell \left( \rho_i^* \cdot u_i^N \right) \). In view of (**), this contradicts \textbf{C05}. Analogously, one obtains a contradiction for \( \lim_{t \to \infty} r_i^N (\rho_t) < r_i^N (\rho) \). Hence, the mapping \( r_i^N \) is continuous.

For all \( i, j, k, \ell \in \Omega \) that are pairwise different, let the mapping \( r_{ij}^{k\ell} : \mathbb{R} \to \mathbb{R} \) be given by
\[
r_{ij}^{k\ell} (\lambda) = \varphi_i \left( \lambda \cdot u_{\{i,j,\ell\}}^{(i,j,k,\ell)} \right) + \varphi_j \left( \lambda \cdot u_{\{i,j,\ell\}}^{(i,j,k,\ell)} \right) \quad \text{for all } \lambda \in \mathbb{R}. \quad (A.7)
\]
The mappings \( r_{ij}^{k\ell} \) have the following properties. By \( \textbf{N} \), we have \( r_{ij}^{k\ell} (0) = 0 \). By \textbf{C06}, the mappings \( r_{ij}^{k\ell} \) are continuous.

\textbf{Claim 7, C07:} For all \( i, j \in \Omega \), \( i \neq j \) such that there exists players \( k, \ell \in \Omega \setminus \{i, j\} \), \( k \neq \ell \) such that \( r_{ij}^{k\ell} \) is unbounded above or below, we have
\[
\varphi \left( -\lambda \cdot u_{\{i,j\}}^{(i,j)} \right) = -\varphi \left( \lambda \cdot u_{\{i,j\}}^{(i,j)} \right) \quad \text{for all } \lambda \in \mathbb{R}.
\]
For all \( i, j, k, \ell \in \Omega \) that are pairwise different and \( \lambda \in \mathbb{R} \), we set \( N := \{i, j, k, \ell\} \) and
\[
\begin{align*}
\lambda_{ijk}^* := & \varphi_i \left( \lambda \cdot u_i^N \right) + \varphi_j \left( \lambda \cdot u_j^N \right) + \varphi_k \left( \lambda \cdot u_k^N \right), \\
\lambda_{ij\ell}^* := & \varphi_i \left( \lambda \cdot u_i^N \right) + \varphi_j \left( \lambda \cdot u_j^N \right) + \varphi_\ell \left( \lambda \cdot u_\ell^N \right), \\
\lambda_{ij}^* := & \varphi_i \left( \lambda \cdot u_i^N \right) + \varphi_j \left( \lambda \cdot u_j^N \right).
\end{align*}
\]
For all \( v \in \mathcal{V} (N) \), we set
\[
\varphi_{i+j} (v) = \varphi_i (v) + \varphi_j (v). \quad (A.11)
\]
For all $\lambda, \lambda_k, \lambda_\ell \in \mathbb{R}$, we have
\[
\varphi_{i+j} \left( -\lambda^*_{ijk} \cdot u^N_{i,j,k} \right) = \varphi_{i+j} \left( \lambda \cdot u^N_i - \lambda^*_{ijk} \cdot u^N_{i,j,k} - \lambda^*_{ij\ell} \cdot u^N_{i,j,\ell} \right) \\
\overset{\text{C03}}{=} \varphi_{i+j} \left( \lambda \cdot u^N_i - \lambda^*_{ijk} \cdot u^N_{i,j,k} \right) + \varphi_{i+j} \left( \lambda \cdot u^N_i - \lambda^*_{ij\ell} \cdot u^N_{i,j,\ell} \right) - \varphi_{i+j} \left( \lambda \cdot u^N_i \right)
\]
\[
\overset{\text{C02}}{=} - \varphi_{i+j} \left( \lambda \cdot u^N_i \right) \\
\overset{(A.10),(A.11)}{=} - \lambda^*_i, \quad \text{(A.12)}
\]
where the first equation drops from
\[
\varphi_\ell \left( \lambda \cdot u^N_i - \lambda^*_{ijk} \cdot u^N_{i,j,k} - \lambda^*_{ij\ell} \cdot u^N_{i,j,\ell} \right) \\
\overset{\text{M}_{i,(A.9),C02}}{=} 0
\]
\[
= \varphi_\ell \left( (\lambda \cdot u^N_i - \lambda^*_{ijk} \cdot u^N_{i,j,k}) \mid_{N \setminus \{i\}} \right) \\
= \varphi_\ell \left( (\lambda \cdot u^N_i - \lambda^*_{ij\ell} \cdot u^N_{i,j,\ell}) \mid_{N \setminus \{j\}} \right),
\]
\[
\varphi_i \left( (\lambda \cdot u^N_i - \lambda^*_{ijk} \cdot u^N_{i,j,k}) \mid_{N \setminus \{i\}} \right) = \varphi_i \left( -\lambda^*_{ijk} \cdot u^N_{i,j,k} \right) \\
\overset{\text{NPO}}{=} \varphi_i \left( -\lambda^*_{ijk} \cdot u^N_{i,j,k} \right),
\]
\[
\varphi_j \left( (\lambda \cdot u^N_i - \lambda^*_{ijk} \cdot u^N_{i,j,k}) \mid_{N \setminus \{j\}} \right) = \varphi_j \left( -\lambda^*_{ijk} \cdot u^N_{i,j,k} \right) \\
\overset{\text{NPO}}{=} \varphi_j \left( -\lambda^*_{ijk} \cdot u^N_{i,j,k} \right),
\]
and $\text{BC}^-$. Further, we have
\[
\varphi_i \left( \lambda \cdot u^N_i \right) + \varphi_i \left( -\lambda^*_{ij} \cdot u^N_{i,j} \right) \overset{\text{C04}}{=} \varphi_i \left( \lambda \cdot u^N_i - \lambda^*_{ij} \cdot u^N_{i,j} \right) \overset{(A.10),C02}{=} 0,
\]
\[
\varphi_i \left( -\lambda^*_{ijk} \cdot u^N_{i,j,k} \right) + \varphi_i \left( \lambda^*_{ij} \cdot u^N_{i,j} \right) \overset{\text{C04}}{=} \varphi_i \left( -\lambda^*_{ijk} \cdot u^N_{i,j,k} - \left( -\lambda^*_{ij} \right) \cdot u^N_{i,j} \right) \overset{(A.10),C02,(A.13)}{=} 0, \quad \text{(A.14)}
\]
and
\[
\varphi_i \left( \lambda \cdot u^N_i \right) + \varphi_i \left( -\lambda^*_{ij} \cdot u^N_{i,j} \right) \overset{(A.8),C02,C04}{=} 0. \quad \text{(A.15)}
\]
Combining the last three equations yields
\[
\varphi_i \left( -\lambda^*_{ij} \cdot u^N_{i,j} \right) \overset{\text{NPO}}{=} \varphi_i \left( -\lambda^*_{ij} \cdot u^N_{i,j} \right) = - \varphi_i \left( \lambda^*_{ij} \cdot u^N_{i,j} \right) \overset{\text{NPO}}{=} - \varphi_i \left( \lambda^*_{ij} \cdot u^N_{i,j} \right). \quad \text{(A.16)}
\]
Suppose that $r_{ij}^{kl}$ is unbounded above. By (A.7) and N, we have (*) $r_{ij}^{kl}(0) = 0$. Fix $\lambda \in \mathbb{R}_+$. By (A.7) and C05, the mapping $r_{ij}^{kl}$ is monotonically increasing. Since $r_{ij}^{kl}$ is unbounded above, there exists some $\rho_\lambda \in \mathbb{R}_+$ such that (**) $\lambda \leq r_{ij}^{kl}(\rho)$. By (A.7) and C06, the mapping $r_{ij}^{kl}$ is continuous. Hence, the intermediate value theorem implies that there exists some $\rho_\lambda \in \mathbb{R}_+$ such that $\lambda = r_{ij}^{kl}(\rho_\lambda)$. By (A.7), (A.10), and (A.16), it is immediate that

$$\varphi_i \left( -\lambda \cdot u_{[i,j]}^{(i,j)} \right) = -\varphi_i \left( \lambda \cdot u_{[i,j]}^{(i,j)} \right) \text{ for all } \lambda \in \mathbb{R}_+.$$

If $\lambda \in \mathbb{R}_-$, the claim drops from putting $-\lambda$ instead of $\lambda$ in the last equation. Analogously, one shows the claim for $r_{ij}^{kl}$ being unbounded below.

**Claim 8, C08**: For all $i, j, k, \ell \in \Omega$ that are pairwise different, we have

$$\varphi_i \left( \lambda \cdot u_{[i,j]}^{(i,j,k,\ell)} \right) = \varphi_i \left( r_{ij}^{kl}(\lambda) \cdot u_{[i,j]}^{(i,j)} \right) \text{ for all } \lambda \in \mathbb{R}.$$

Set $N := \{i, j, k, \ell\}$. We have

$$\varphi_i \left( \lambda \cdot u_N^{ij} \right) = \varphi_i \left( \lambda \cdot u_{[i,j]}^{ij} \right) = \varphi_i \left( r_{ij}^{kl}(\lambda) \cdot u_{[i,j]}^{ij} \right) = \varphi_i \left( r_{ij}^{kl}(\lambda) \cdot u_{[i,j]}^{ij} \right),$$

which proves the claim.

**Claim 9, C09**: For all $i, j \in \Omega$, $i \neq j$ such that there exists players $k, \ell \in \Omega \setminus \{i, j\}$, $k \neq \ell$ such that $r_{ij}^{kl}$ is unbounded above or below, we have

$$\varphi \left( \lambda \cdot u_{[i,j]}^{ij} \right) = \varphi \left( r_{ij}^{kl}(\lambda) \cdot u_{[i,j]}^{ij} \right) \text{ for all } \lambda \in \mathbb{R}.$$

Let $i, j, k, \ell \in \Omega$ be as in the claim and set $N := \{i, j, k, \ell\}$. We show the claim for $r_{ij}^{kl}$ being unbounded above. Analogously, one can handle the case of $r_{ij}^{kl}$ being unbounded below.

For $\lambda = 0$, the claim drops from N. Set

$$\tilde{\beta} := \varphi_i \left( u_{[i,j,k]}^{N} \right) + \varphi_j \left( u_{[i,j,k]}^{N} \right) > 0 \quad \text{E.C01}.$$

Fix $\alpha \in \mathbb{R}_{++}$ and $\beta \in \left(0, \tilde{\beta} \right]$. By C05 and C06, the mapping and $r_{ij}^{kl}$ is monotonically increasing and continuous. Hence, by the intermediate value theorem, since $r_{ij}^{kl}$ is unbounded above, and $r_{ij}^{kl}(0) = 0$, there exist $\alpha^*, \beta^*, \gamma^* \in \mathbb{R}_{++}$ such that

$$r_{ij}^{kl}(\alpha^*) = \alpha, \quad r_{ij}^{kl}(\beta^*) = \beta, \quad \text{and} \quad r_{ij}^{kl}(\gamma^*) = \alpha + \beta. \quad \text{(A.18)}$$

Let the mapping $r_{ijk}^\ell : \mathbb{R}_{++} \to \mathbb{R}$ given by

$$r_{ijk}^\ell(\lambda) = \varphi_i \left( \lambda \cdot u_{[i,j,k]}^{N} \right) + \varphi_j \left( \lambda \cdot u_{[i,j,k]}^{N} \right) \text{ for all } \lambda \in \mathbb{R}. \quad \text{(A.19)}$$
By C05 and C06, the mapping and $r_{ij}^t$ is monotonically increasing and continuous. Hence, by the intermediate value theorem, since $r_{ij}^t(0) = 0$, and

$$r_{ij}^t(1) \overset{(A.17),(A.19)}{=} \beta,$$

there exists $\beta^{**} \in \mathbb{R}_{++}$ such that

$$\varphi_i (\beta^{**} \cdot u_{i,j,k}^N) + \varphi_j (\beta^{**} \cdot u_{i,j,k}^N) = \beta.$$  \hspace{1cm} (A.20)

Let $i \in \{i, j\}$. We have

$$\varphi_i (\alpha^* \cdot u_N^N + \beta^{**} \cdot u_{i,j,k}^N) \overset{C04}{=} \varphi_i (\alpha^* \cdot u_N^N) + \varphi_i (\beta^{**} \cdot u_{i,j,k}^N) \overset{(A.20)}{=} \varphi_i (\alpha^* \cdot u_N^N) + \varphi_i (\beta \cdot u_{i,j}^{ij}) \overset{(A.18), C08}{=} \varphi_i (\alpha^* \cdot u_N^N) + \varphi_i (\beta \cdot u_{i,j}^{ij}) + \varphi_i (\alpha \cdot u_{i,j}^{ij}) + \varphi_i (\beta \cdot u_{i,j}^{ij}).$$  \hspace{1cm} (A.21)

and therefore

$$\varphi_i (\alpha^* \cdot u_N^N + \beta^{**} \cdot u_{i,j,k}^N) + \varphi_j (\alpha^* \cdot u_N^N + \beta^{**} \cdot u_{i,j,k}^N) \overset{E}{=} \alpha + \beta.$$  \hspace{1cm} (A.22)

Thus, we thus have

$$0 \overset{C02,(A.22)}{=} \varphi_i (\alpha^* \cdot u_N^N + \beta^{**} \cdot u_{i,j,k}^N) - (\alpha + \beta) \cdot u_{i,j}^{ij}) \overset{C04}{=} \varphi_i (\alpha^* \cdot u_N^N + \beta^{**} \cdot u_{i,j,k}^N) + \varphi_i (-(\alpha + \beta) \cdot u_{i,j}^{ij}) \overset{C07}{=} \varphi_i (\alpha^* \cdot u_N^N + \beta^{**} \cdot u_{i,j,k}^N) - \varphi_i ((\alpha + \beta) \cdot u_{i,j}^{ij})$$  \hspace{1cm} (A.23)

and therefore

$$\varphi_i (\alpha + \beta) \cdot u_{i,j}^{ij})) \overset{(A.21),(A.23),NPO}{=} \varphi_i (\alpha \cdot u_{i,j}^{ij}) \overset{(A.24)}{=} \varphi_i (\beta \cdot u_{i,j}^{ij}).$$  \hspace{1cm} (A.24)

By (A.24), the mappings $f_i: \mathbb{R}_+ \rightarrow \mathbb{R}$ given by

$$f_i(\lambda) := \varphi_i (\lambda \cdot u_{i,j}^{ij}) \quad \text{for all } \lambda \in \mathbb{R}_+$$  \hspace{1cm} (A.25)

are additive. By C05 and C06, these mappings are monotonically increasing and continuous. Hence by Aczél (1966, Corollary 9), for example, we have $f_i(\lambda) = \lambda \cdot f_i(1)$ for all $\lambda \in \mathbb{R}_+$, i.e.,

$$\varphi_i (\lambda \cdot u_{i,j}^{ij}) = \lambda \cdot \varphi_i (u_{i,j}^{ij}) \quad \text{for all } \lambda \in \mathbb{R}_+.$$  

Invoking C07 concludes the proof of the claim.

Claim 10, C10: For all $i, j \in \Omega$, $i \neq j$ and $\lambda \in \mathbb{R}$, we have

$$\varphi (\lambda \cdot u_{i,j}^{ij}) = \lambda \cdot \varphi (u_{i,j}^{ij}).$$  

15
Let $i, j, k, \ell \in \mathcal{U}$ be pairwise different and set $N := \{i, j, k, \ell\}$. We show that $r_{ij}^{kl}$ is unbounded above or below. The claim then drops from C09. Suppose (*) $r_{ij}^{kl}$ is bounded above and below. By E, C01, and (A.7), either $r_{ik}^{jl}$ or $r_{il}^{jk}$ must be unbounded above or below. W.l.o.g., let $r_{ik}^{jl}$ be unbounded above or below. For all $\lambda \in \mathbb{R}$, we obtain

$$\varphi_i \left( \lambda \cdot u_N^i \right) \overset{(A.7)\cdot C08}{=} \varphi_i \left( r_{ik}^{jl} (\lambda) \cdot u_{\{i,k\}}^{i,j} \right) \overset{C09}{=} r_{ik}^{jl} (\lambda) \cdot \varphi_i \left( u_{\{i,k\}}^{i,j} \right).$$  \hspace{1cm} (A.26)

By E and C01, we have $\varphi_i \left( u_{\{i,k\}}^{i,j} \right) > 0$. Hence, by (A.26) and $r_{ik}^{jl}$ being unbounded above or below, the expression $\varphi_i \left( \lambda \cdot u_N^i \right)$ is unbounded above or below in $\lambda \in \mathbb{R}$. By C01, all payoffs for $\lambda \cdot u_N^i$ have the same sign. Hence, by (A.7), the mapping $r_{ij}^{kl}$ is unbounded above or below, contradicting assumption (*) on $r_{ij}^{kl}$.

**Claim 11, C11:** For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i, j, k \in N$ such that $i, j, k$ are different and pairwise mutually dependent in $v$, we have

$$\varphi_i (v) = \varphi_i \left( (\varphi_i (v) + \varphi_j (v)) \cdot u_{\{i,j\}}^{i,j} \right).$$

We obtain

$$0 \overset{C02}{=} \varphi_i \left( v - (\varphi_i (v) + \varphi_j (v)) \cdot u_{\{i,j\}}^{N} \right) \overset{C04}{=} \varphi_i (v) + \varphi_i \left( - (\varphi_i (v) + \varphi_j (v)) \cdot u_{\{i,j\}}^{N} \right)$$

$$\overset{NPO}{=} \varphi_i (v) + \varphi_i \left( - (\varphi_i (v) + \varphi_j (v)) \cdot u_{\{i,j\}}^{\{i,j\}} \right) \overset{C10}{=} \varphi_i (v) - \varphi_i \left( (\varphi_i (v) + \varphi_j (v)) \cdot u_{\{i,j\}}^{\{i,j\}} \right),$$

which proves the claim.

Construct $w^\varphi \in \mathbb{R}_{++}^\mathcal{U}$ as follows. Fix $i \in \mathcal{U}$ and set

$$w_i^\varphi := 1 \quad \text{and} \quad w_j^\varphi := \frac{\varphi_j \left( u_{\{i,j\}}^{\{i,j\}} \right)}{\varphi_i \left( u_{\{i,j\}}^{\{i,j\}} \right)} \quad \text{for all } j \in \mathcal{U} \setminus \{i\}. \hspace{1cm} (A.27)$$

By E and C1, we have $\varphi_i \left( u_{\{i,j\}}^{\{i,j\}} \right) > 0$ and $\varphi_j \left( u_{\{i,j\}}^{\{i,j\}} \right) > 0$ entailing that $w^\varphi$ is well-defined and, indeed, $w^\varphi \in \mathbb{R}_{++}^\mathcal{U}$. In the following, we show that $\varphi (v) = \mathbb{S}h_{w^\varphi} (v)$ for all $v \in \mathbb{V}(N)$, $N \in \mathcal{N}$.

**Claim 12, C12:** For all $i, j \in N$, $i \neq j$, we have

$$\frac{\varphi_i \left( u_{\{i,j\}}^{\{i,j\}} \right)}{\varphi_j \left( u_{\{i,j\}}^{\{i,j\}} \right)} = \frac{w_i^\varphi}{w_j^\varphi}. \hspace{1cm} (A.28)$$
By \( E \) and \( C1 \), we have \( \varphi_i (u_{(i,j)}^{(i,j)}) > 0 \) and \( \varphi_j (u_{(i,j)}^{(i,j)}) > 0 \). If \( i = j \) or \( j = i \), then the claim drops from (A.27) and (4). For \( i, j \in \mathcal{U} \setminus \{i\} \), we obtain

\[
\frac{\varphi_i (u_{(i,j)}^{(i,j)})}{\varphi_j (u_{(i,j)}^{(i,j)})} = \frac{\varphi_i (u_{(i,j)}^{(i,j)})}{\varphi_j (u_{(i,j)}^{(i,j)})} = \frac{\varphi_i (u_{(i,j)}^{(i,j)})}{\varphi_j (u_{(i,j)}^{(i,j)})} \overset{C10,C11}{=} \frac{\varphi_i (u_{(i,j)}^{(i,j)})}{\varphi_j (u_{(i,j)}^{(i,j)})} \overset{C10,C11}{=} \frac{\varphi_i (u_{(i,j)}^{(i,j)})}{\varphi_j (u_{(i,j)}^{(i,j)})} \overset{(A.27)}{=} \frac{w_i^\varphi}{w_j^\varphi}.
\]

Claim 13, C13: For all \( N \in \mathcal{N} \), \( v \in \mathcal{V}(N) \), and \( i, j, k \in N \) such that \( i, j, k \) are different and pairwise mutually dependent in \( v \), we have

\[
\varphi_i (v) \cdot w_j^\varphi = \varphi_j (v) \cdot w_i^\varphi.
\]

By \( C01 \), the claim holds true if \( \varphi_i (v) = 0 \). If \( \varphi_i (v) \neq 0 \), then \( C01 \) entails \( \varphi_j (v) \neq 0 \) and \( \varphi_i (v) + \varphi_j (v) \neq 0 \). Hence, we obtain

\[
\frac{\varphi_i (v)}{\varphi_j (v)} = \frac{\varphi_i ((\varphi_i (v) + \varphi_j (v)) \cdot u_{(i,j)}^{(i,j)})}{\varphi_j ((\varphi_i (v) + \varphi_j (v)) \cdot u_{(i,j)}^{(i,j)})} \overset{C10}{=} \frac{\varphi_i (u_{(i,j)}^{(i,j)})}{\varphi_j (u_{(i,j)}^{(i,j)})} \overset{C12,(A.27)}{=} \frac{w_i^\varphi}{w_j^\varphi}.
\] (A.29)

Now, we are ready to show \( \varphi = \text{Sh}^{w^\varphi} \) along the lines of Young (1985). For \( N \in \mathcal{N} \) such that \( |N| = 1 \), the claim is immediate from \( E \). Fix now \( N \in \mathcal{N} \) such that \( |N| > 1 \). We proceed by induction on \( |T(v)| \) (see (A.1)).

Induction basis: If \( |T(v)| = 0 \), then \( v = 0^N \). By \( N \), we have \( \varphi (v) = \text{Sh}^{w^\varphi} (v) \).

Induction hypothesis (IH): Assume now that \( \varphi (v) = \text{Sh}^{w^\varphi} (v) \) for all \( v \in \mathcal{V}(N) \) such that \( |T(v)| \leq t, t \in \mathbb{N} \).

Induction step: Let \( v \in \mathcal{V}(N) \) be such that \( |T(v)| = t + 1 \). Set

\[
P(v) := \{ \ell \in N \mid \ell \in T \text{ for all } T \in T(v) \}.
\] (A.30)

For \( i \in N \setminus P(v) \) and \( T \in T(v) \) such that \( i \notin T \), we obtain

\[
\varphi_i (v) \overset{M}{=} \varphi_i (v - \lambda_T (v) \cdot u_T^N) \overset{IH}{=} \text{Sh}_i^{w^\varphi} (v - \lambda_T (v) \cdot u_T^N) \overset{M}{=} \text{Sh}_i^{w^\varphi} (v)
\] (A.31)

and

\[
\sum_{\ell \in P(v)} \varphi_\ell (v) \overset{E}{=} \sum_{\ell \in P(v)} \text{Sh}_\ell^{w^\varphi} (v).
\] (A.32)

If \( |P(v)| \leq 1 \), then \( E \) and (A.31) imply \( \varphi (v) = \text{Sh}^{w^\varphi} (v) \).

Let now \( |P(v)| \geq 3 \). Fix \( i \in P(v) \). By (A.1) and (A.30), any two players in \( P(v) \) are mutually dependent in \( v \). Hence, we have

\[
\varphi_i (v) \cdot w_i^\varphi \overset{C13}{=} \varphi_i (v) \cdot w_i^\varphi \text{ for all } i \in P(v) \setminus \{i\}.
\] (A.33)
The system of equations (A.32) and (A.33) has a unique solution, which must be
\[
(\varphi_i (v))_{i \in P(v)} = (\text{Sh}^{w^r}_i (v))_{i \in P(v)}.
\]
It is clear that these payoffs satisfy (A.32). By (4), they also meet (A.33). In view of (A.31), we obtain \( \varphi (v) = \text{Sh}^{w^r} (v) \).

Let now \(|P(v)| = 2\). For all \( C \subseteq T(v) \), set
\[
v_C := \sum_{T \in C} \lambda_T (v) \cdot u_T^N
\]
and choose \( C^* \subseteq T(v) \) such that \(|P(v_{C^*})| > 2\) and \(|P(v_C)| = 2\) for all \( C \subseteq T(v) \) such that \( C^* \subseteq C \). If \( C^* = \emptyset \), then \( v = \lambda \cdot u^N_{P(v)} \) for some \( \lambda \in \mathbb{R} \setminus \{0\} \). By E, N, NPO, C12, and (4), \( \varphi (v) = \text{Sh}^{w^r} (v) \). If \( C^* \neq \emptyset \), we have
\[
\varphi_i (v) = \varphi_i (v_{C^*} + v_{C \setminus C^*}) \overset{\text{C04}}{=} \varphi_i (v_{C^*}) + \varphi_i (v_{C \setminus C^*})
\]
\[
\overset{\text{IH}}{=} \text{Sh}^{w^r}_i (v_{C^*}) + \text{Sh}^{w^r}_i (v_{C \setminus C^*}) = \text{Sh}^{w^r}_i (v_{C^*} + v_{C \setminus C^*}) = \text{Sh}^{w^r}_i (v)
\]
for all \( i \in P(v) \), which concludes the proof.

References


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